Perfect Squared
Klein Bottle *Myths*

by Geoffrey H. Morley, January 2014
(Enhanced version of my MathsJam 2013 talk.)

A PERFECT squaring is a tiling by squares (two or more) of distinct sizes.

=> "Every tiling of a Mobius strip can be glued along its edge (there is only one) to give a tiling of the Klein bottle, and there are no other ways to tile the Klein bottle with six or fewer [distinct] tiles." (Stewart, 1997 and 2004)
"Any squaring of the Mobius strip gives a squaring of the Klein bottle. For 6 or fewer tiles these are the only ones." (Gale, 1998)
- NOT TRUE!
There are many other ways to tile a Klein bottle with six or fewer distinct square tiles as I shall demonstrate!

=> "Nobody knows whether this remains true for seven or eight tiles." (Stewart, 1997 and 2004)
"in the case of 7 or 8 tiles, this is not known." (Gale, 1998)
- POSSIBLY TRUE THEN but many tilings with 7 or 8 tiles are now known.

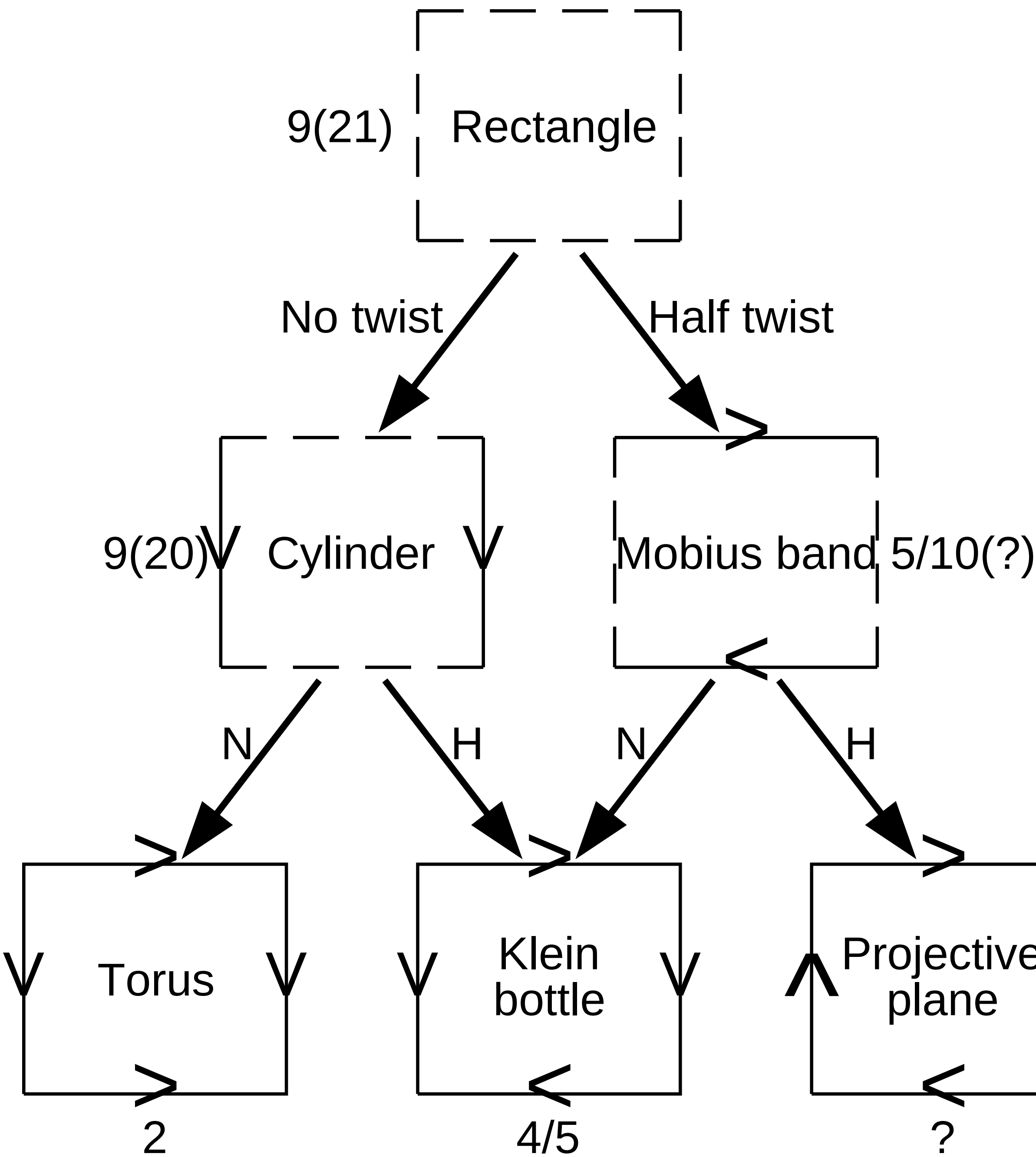
=> "it is not known whether there are tilings of the Klein bottle in which the tiles need not be parallel to the sides of the big square [or rectangle]." (Gale, 1998)
- POSSIBLY TRUE THEN but such tilings are now known with as few as four distinct squares.

Fewest Squares for a Perfect Faultfree Squaring

Opposite edges are glued, or identified in the imagination, with the direction of their arrows coinciding. Broken lines remain edges. Joining two opposite unglued edges, with or without a half twist, gives another surface. A FAULTFREE tiling lacks fault lines (glued edges that do not intersect any tiles).

Each number after a slash is the fewest squares for a perfect tiling when a shorter edge was given a half twist. 21 and 20 are the fewest squares in a perfect squared square or perfect squared square-cylinder respectively.

In every squared Mobius band or Klein bottle depicted later, the edges to be joined after a half twist are horizontal.

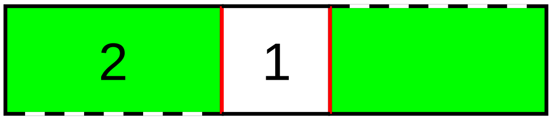


Perfect Squared Mobius Bands with the fewest squares

Nonfaultfree

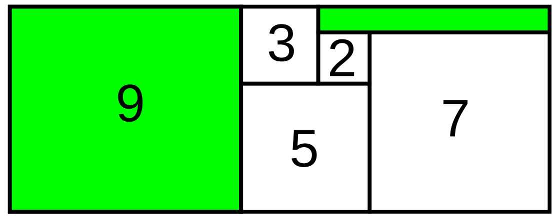
See http://mathworld.wolfram.com/ MoebiusStripDissection.html

Cutting the red fault line yields two interlinked bands.



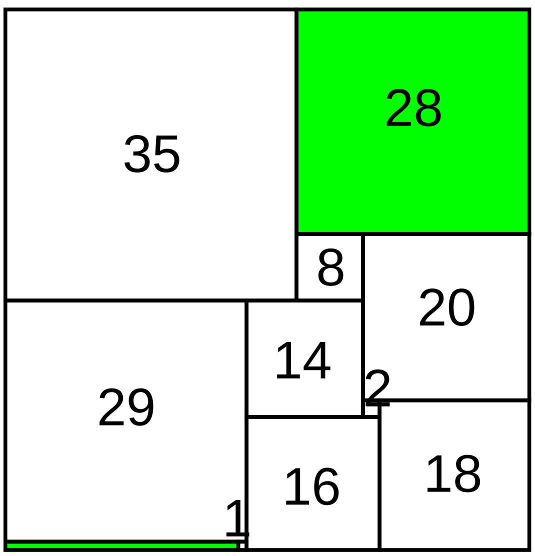
2: 5 x 1 (SJC)

Faultfree

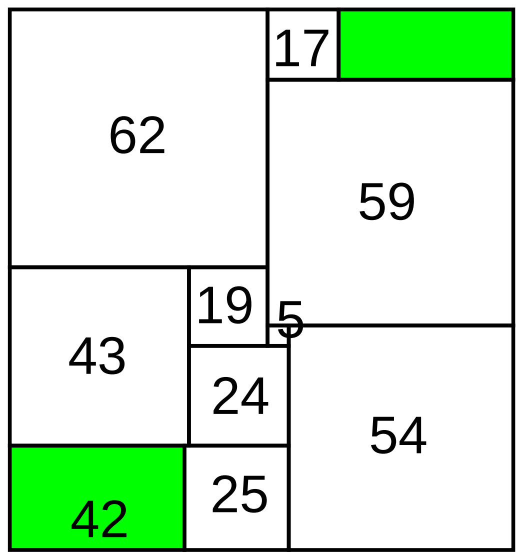


5: 21 x 8 (SJC)

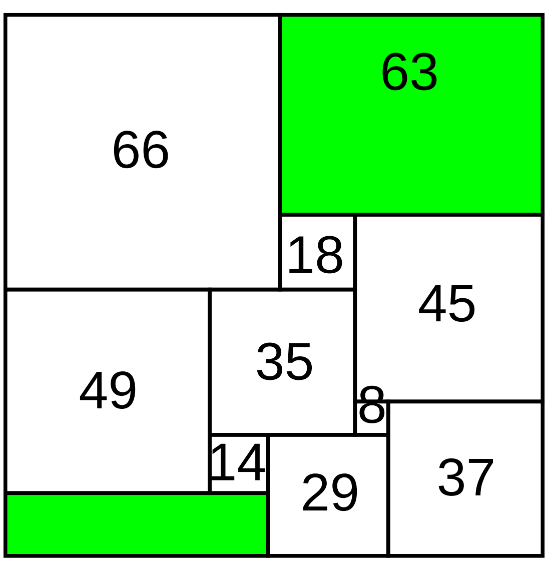
Faultfree with the shorter edge given a half twist



10: 63 x 65 (GHM)



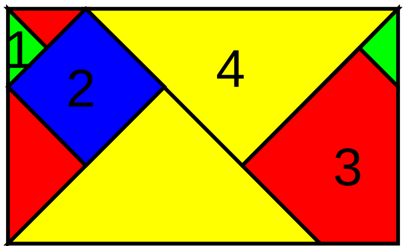
10: 121 x 130 (GHM)



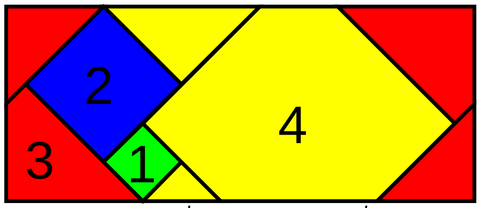
10: 129 x 130 (GHM)

I conjecture that there are no other such tilings with fewer than 11 squares. Martin Gardner (Barr, 1964), and probably others, observed that a rectangular strip too wide to be simply twisted and joined can be crimped into one narrow enough that can with a sufficient even number of equally spaced parallel folds. With two folds, for example, the cross section of the strip is an N which, being still an N after a half twist, is compatible with joining the edges.

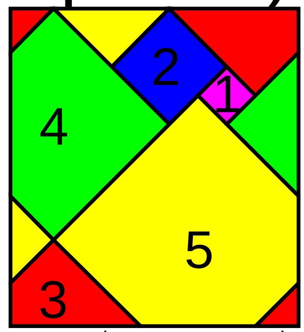
Faultfree Perfect Squared Klein Bottles with up to 6 squares (complete?)



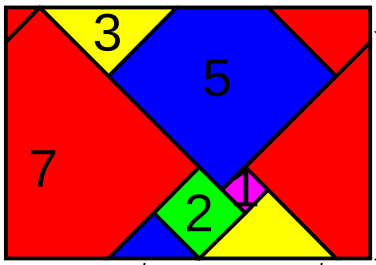
4: $5\sqrt{2} \times 3\sqrt{2}$



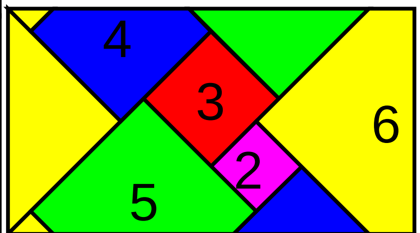
4: $6\sqrt{2} \times 2.5\sqrt{2}$



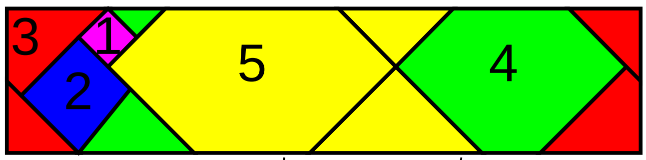
5: $5\sqrt{2} \times 5.5\sqrt{2}$



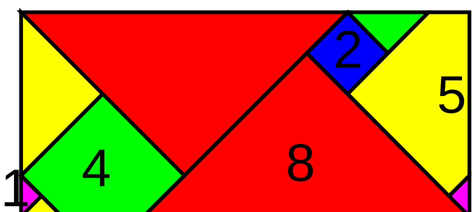
5: $8\sqrt{2} \times 5.5\sqrt{2}$



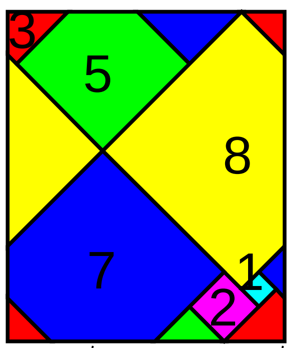
5: $9\sqrt{2} \times 5\sqrt{2}$



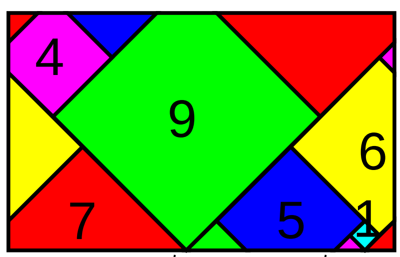
5: $11\sqrt{2} \times 2.5\sqrt{2}$



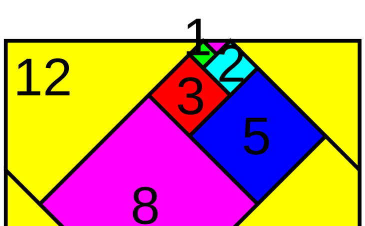
5: $11\sqrt{2} \times 5\sqrt{2}$



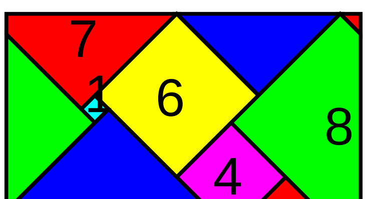
6: $8\sqrt{2} \times 9.5\sqrt{2}$



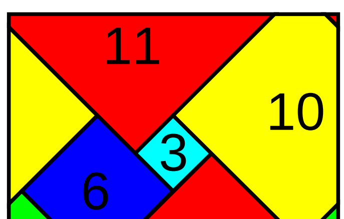
6: $13\sqrt{2} \times 8\sqrt{2}$



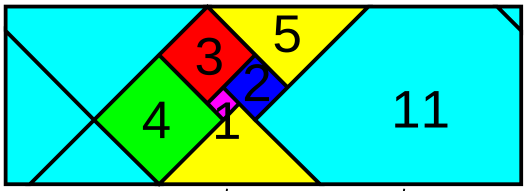
6: $13\sqrt{2} \times 9.5\sqrt{2}$ (1/2)



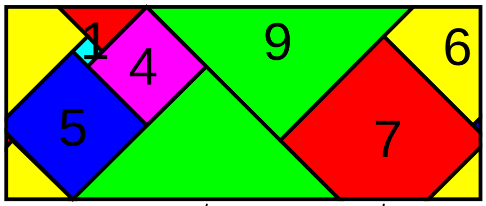
6: $13\sqrt{2} \times 9.5\sqrt{2}$ (2/2)



6: $13\sqrt{2} \times 11\sqrt{2}$

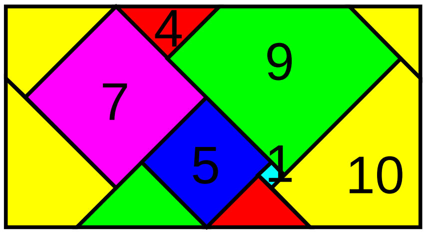


6: $16\sqrt{2} \times 5.5\sqrt{2}$

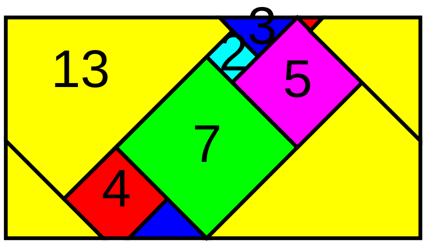


6: $16\sqrt{2} \times 6.5\sqrt{2}$

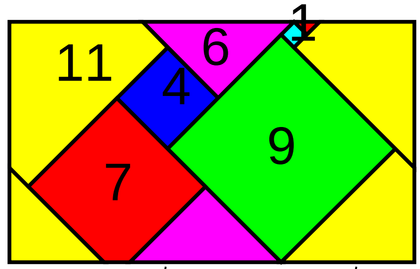
Faultfree Perfect Squared Klein Bottles with up to 6 squares (complete?) (cont.)



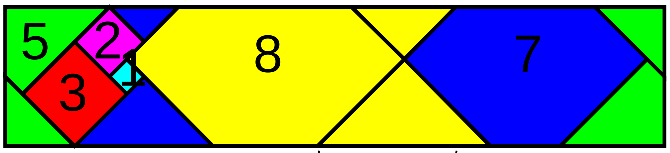
6: $16\sqrt{2} \times 8.5\sqrt{2}$ (1/2)



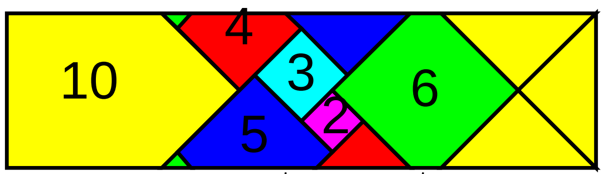
6: $16\sqrt{2} \times 8.5\sqrt{2}$ (2/2)



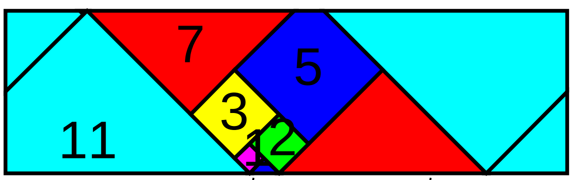
6: $16\sqrt{2} \times 9.5\sqrt{2}$



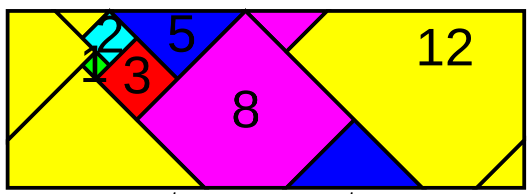
6: $19\sqrt{2} \times 4\sqrt{2}$



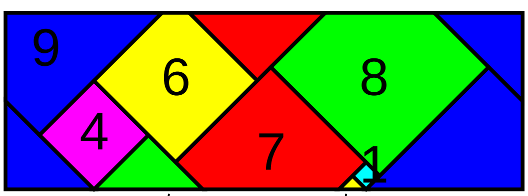
6: $19\sqrt{2} \times 5\sqrt{2}$



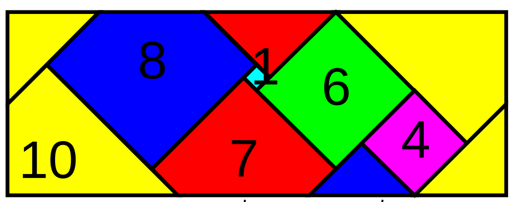
6: $19\sqrt{2} \times 5.5\sqrt{2}$



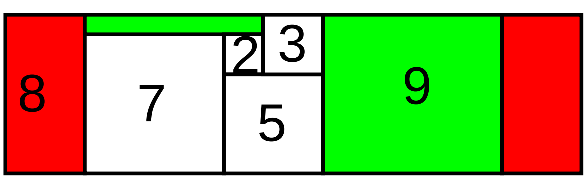
6: $19\sqrt{2} \times 6.5\sqrt{2}$ (1/2)



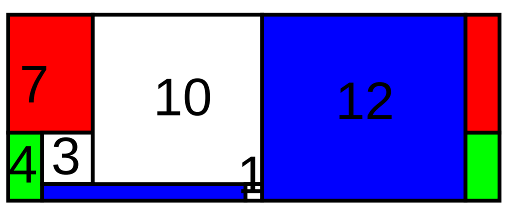
6: $19\sqrt{2} \times 6.5\sqrt{2}$ (2/2)



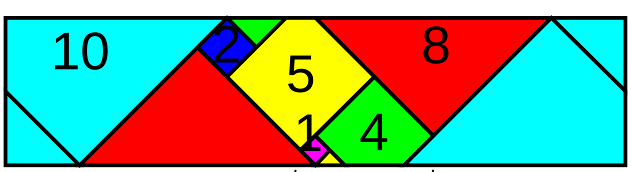
6: $19\sqrt{2} \times 7\sqrt{2}$



6: 29×8

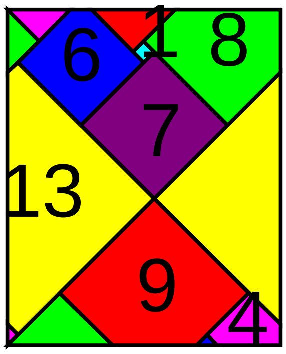


6: 29×11

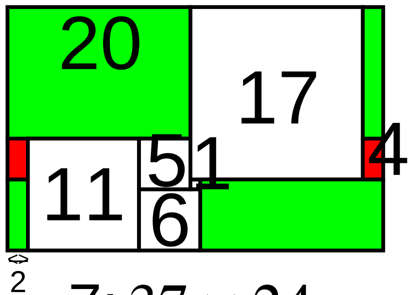


6: $21\sqrt{2} \times 5\sqrt{2}$

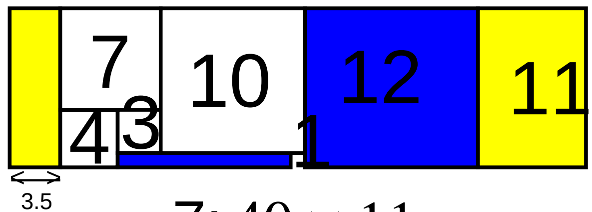
Faultfree Perfect Squared Klein Bottles
- examples with 7, 8 and 22 squares



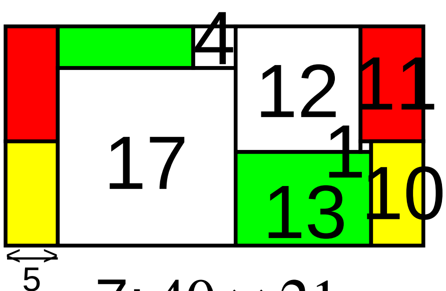
7: $13\sqrt{2} \times 16\sqrt{2}$



7: 37×24

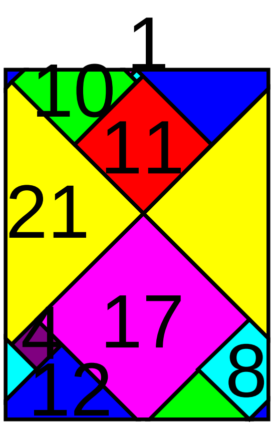


7: 40×11

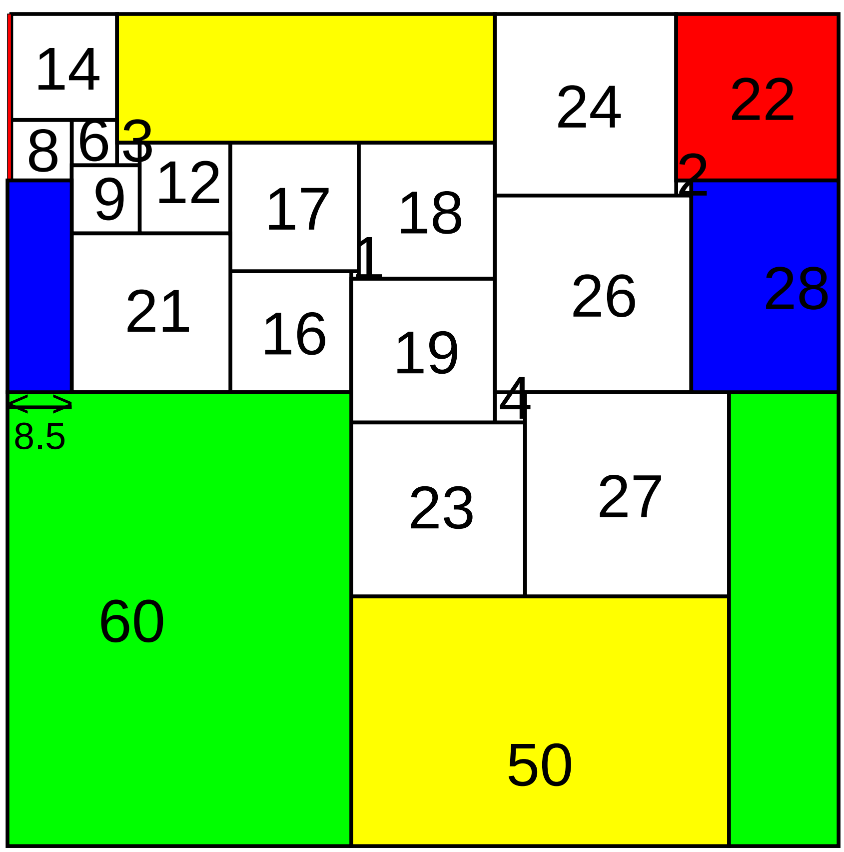


7: 40×21

There are at least 49 faultfree perfect squared Klein bottles with 7 squares.



8: $21\sqrt{2} \times 28\sqrt{2}$



22: 110×110 (THW&GHM)

Perfect Squared
Klein Bottle *Myths*

PROBLEMS

1. Chapman (1993) claimed to have found every perfect squared Mobius band with six square tiles, including one with tile sides 2, 3, 5, 7, 8, 9 in which the two even-sided tiles are nonadjacent (Figure 12(b) on page 480).
   But he failed to mention another arrangement of these tiles in which the even-sided tiles are adjacent.
   Find both tilings.

2. Find further faultfree perfect squared Klein bottles with as few squares as possible, both with tile sides parallel to the sides of the rectangle and at 45 degrees.
   (Why no other angle?)

3. How can all faultfree perfect squared Mobius bands or Klein bottles with a given number of squares be enumerated?
   And what about tilings of the projective plane?

REFERENCES

Barr, Stephen (1964), Experiments in Topology, New York: Thomas Y. Crowell Company, pp. 48, 200-201.

Chapman, S. J. (November 1993), The dissection of rectangles, cylinders, tori, and Mobius bands into squares, Duke Mathematical Journal, Vol.72, Issue 2, pp. 467-485.

Gale, David (1998), Tracking the Automatic Ant, Springer-Verlag, New York, p. 58.

Stewart, Ian (July 1997), Squaring the Square, Scientific American, pp. 74-76.

Stewart, Ian (2004), Math Hysteria, Oxford University Press, pp. 152-153. [Same remarks as in the 1997 article.]