Compound Perfect Squared Squares of the Order Twenties

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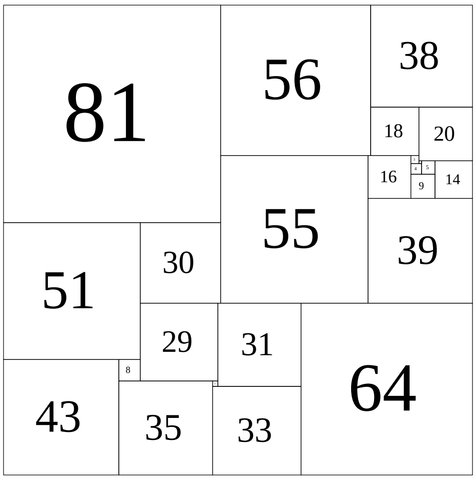
February 10, 2013

Abstract

P. J. Federico used the term *low-order* for perfect squared squares with at least 28 squares in their dissection. In 2010 low-order compound perfect squared squares (CPSSs) were completely enumerated. There are a total of 207 low-order CPSSs found in orders 24 to 28. In 2012 CPSSs were enumerated to order 29. A total of 413 CPSSs were found in order 29. All together there are 620 CPSS, (up to symmetries of the square and its squared subrectangles), in the order twenties, from order 24 to 29.

Contents



24 : 175a (1/4) THW (1948)

Figure 1: T. H. Willcocks order 24 CPSS, side 175, 1 of 4 isomers, (1948): bouwkampcode; (81,56,38)(18,20)(55,16)(1,5,14)(4)(39)(51,30)(29,31,64)(43,8)(35,2)(33) tablecode; 24 175 175 81 56 38 18 20 55 16 1 5 14 4 39 51 30 29 31 64 43 8 35 2 33

1 Definitions and Terminology

1.1 Squared rectangles and squared squares

A *squared rectangle* is a rectangle dissected into a finite number, two or more, of squares, called the *elements* of the dissection. If no two of these squares have the same size the squared rectangle is called *perfect*, otherwise it is *imperfect*. The *order* of a squared rectangle is the number of constituent squares. The case in which the squared rectangle is itself a square is called a *squared square*. The dissection is *simple* if it contains no smaller squared rectangle, otherwise it is *compound*.

A squared square which is both compound and perfect is called a *compound perfect squared square* (CPSS).

By a result of Dehn[17], a rectangle can be tiled by a finite number of squares if and only if the rectangle has commensurable sides. From commensurability it follows that the squared rectangles sides and elements can all be given in integers. Since the first perfect squared rectangles were published by Z. Moroń[40] two conventions have been followed; expressing the rectangle sides and elements in integers without any common divisor (unless some reason requires otherwise), and writing the length of the side of an element centered inside that square in illustrations.

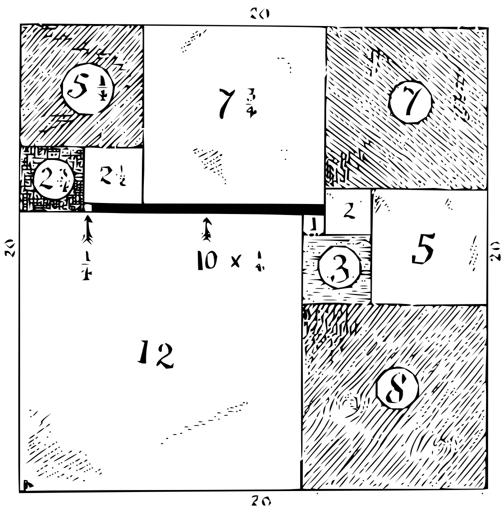


Figure 2: Lady Isabel's Casket solution

1.2 Isomers of compound perfect squared squares

A CPSS can be rotated and reflected in 8 ways creating a isomorphism class of equivalent dissections, we call this the CPSS class. Any smaller squared rectangles within the CPSS can also be independently rotated and reflected creating an additional isomorphism class of CPSSs with equivalent elements, we call this the CPSS isomer class. We say each member of that class is an isomer of the CPSS. We allow a single CPSS representative to stand for all the members of the CPSS class and the CPSS isomer class. Sometimes the isomer count is also given, that is, the number of members of the isomer class of a CPSS. See subsection 3.7 for further details. The method of selecting the CPSS representative from the CPSS isomers is given in subsection 3.8.

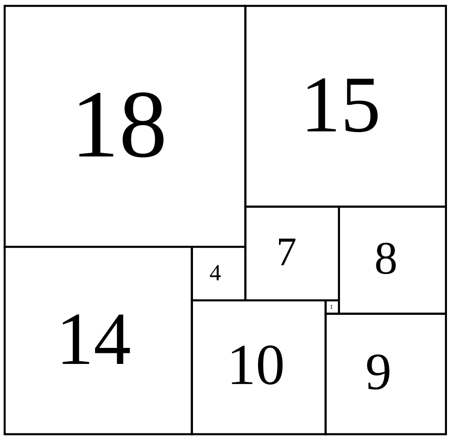
2 History of CPSS Discoveries: 1902 - 2013

1902

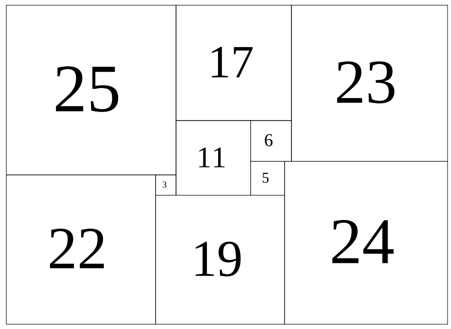
H.E. Dudeney published a puzzle called Lady Isabel's Casket that concerns the dissection of a square into different sized squares and a rectangle. According to David Singmaster[43] 'Lady Isabel's Casket' appeared first in Strand Magazine January 1902 and is the first published reference dealing with the dissection of a square into smaller different sized squares. 'Lady Isabel's Casket' was also published in The Canterbury Puzzles[18] in 1907. (See Figure 2).

1903

Max Dehn studied the squaring problem[17] and proved; A rectangle can be squared if and only if its sides are commensurable (in rational proportion, both being integer multiples of the same quantity). He also proved that if a rectangle can be squared



9 : 33 x 32, Rectangle I



10 : 65 x 47, Rectangle II

Figure 3: Z. Moroń's Rectangle I, Rectangle II

then there are infinitely many perfect squarings. This result has been generalised and extended, see Wagon [54].

1907-1917

S. Loyd published The Patch Quilt Puzzle; *A square quilt made of 169 square patches of the same size is to be divided into the smallest number of square pieces by cutting along lattice lines, find the sizes of the squares.*. The answer, which is unique, is composed of 11 squares with sides 1,1,2,2,2,3,3,4,6,6,7 within a square of 13. It is imperfect and compound. Gardner states that this problem first appeared in 1907 in a puzzle magazine edited by Sam Loyd. David Singmaster credits Loyd with publishing Our Puzzle Magazine in 1907 - 08. This puzzle also appeared in a publication by Henry Dudeney as Mrs Perkin's quilt[56, 6], Problem 173 in Amusements in Mathematics[19] (1917).

1925

Zbigniew Moroń published a paper[40], where he gave the first examples of rectangles divided into unequal squares. Rectangle I is 33 x 32 in size and is divided into 9 unequal squares. Rectangle II is 65 x 47 and has ten squares. See Figure 3.

Moroń asked the question "For what squares is it possible to dissect them into squares?". He then observes "if there exists a rectangle (of different sides) for which there are two dissections R1 and R2 such that; in neither of these dissections does there appear a square equal to to the smaller side of the rectangle and, each square of dissection R1 is different from each square in dissection R2, then the square is dissected into squares, all different."

1930

Kraitchik[33] published the proposition, communicated to him by the Russian mathematician N. N. Lusin, that it was not possible to divide a square into a finite number of different squares.

1931-1932

A Japanese mathematician Michio Abe, published 2 papers[1, 2] on the problem. He produced over 600 squared rectangles, in his second paper he gave simple perfect squared rectangle with sides 195 x 191 and showed how it can be used to construct an infinite series of compound squared rectangles with the ratio of sides approaching one in the limit.

1937-1939

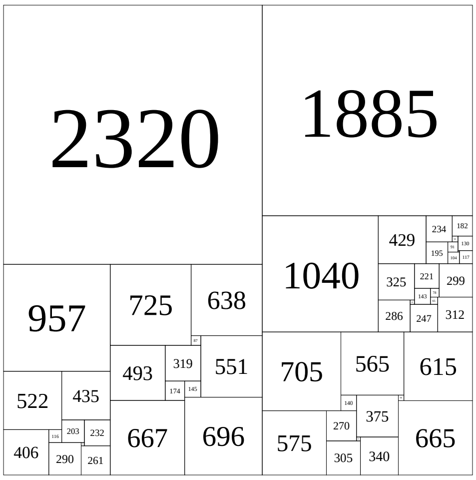
A number of publications on the problem of squaring the square appeared in Germany by Jaremkewycz, Mahrenholz, Sprague[30], A. Stöhr[48], H. Reichardt and H. Toepkin[49, 41]. Following these publications, R.P. Sprague published[45] his solution to the problem of squaring the square. Sprague constructed his perfect solution using several copies of various sizes of Z. Moroń's Rectangle I (33x32), Rectangle II (65x47) and a third order 12 simple perfect rectangle (377x256) and five other elemental squares to create an order 55, compound perfect squared square (CPSS) with side 4205 (Figure 4).

In the same year the minutes of two different meetings of the Trinity Mathematical Society at Trinity College, Cambridge University announce the discoveries of some perfect squared squares. On 13 March 1939, the minutes[47] record A. Stone's lecture: "Squaring the Square" where he announces R. Brooks's squared square with 39 elements, a side of 4639 and containing a perfect subrectangle (a CPSS).

1940

Four undergraduates at Trinity College Cambridge, R.L. Brooks, C.A.B. Smith, A.H. Stone and W.T.Tutte published the classic paper *The Dissection of Rectangles into Squares*[42]. They published an empirically constructed order 26 CPSS, side 608, (attributed later to Tutte) and referred the use of 2 order 13 SPSRs with different elements to construct a Moroń R1, R2 dissection CPSS of order 28 (attributed later to A.H. Stone) with side 1015[36](Figure 5) and mentioned a second CPSS, also with side 1015. By associating a squared rectangle with a certain type of electrical network they developed an extensive theory of squared rectangles which combined the theory of planar graphs and of electrical networks. By exploiting rotational symmetry in a 3-pole electrical network they developed methods for creating perfect squared squares in order 30's and above (both CPSSs and SPSSs). The theory was generalized to a variety of non-rectangular dissections and in particular to triangled equilateral triangles in later papers[51].

Later in the same year Tutte published[36] his solution to problem E401 which included the previously mentioned second CPSS of order 28, also with a side of



55 : 4205 x 4205, R.P.S. Sprague (1939)

Figure 4: Sprague's Order 55 CPSS

1015 but almost completely different to Stone's 1015. When compared element by element these two CPSSs have only 2 elements in common.

1946-48

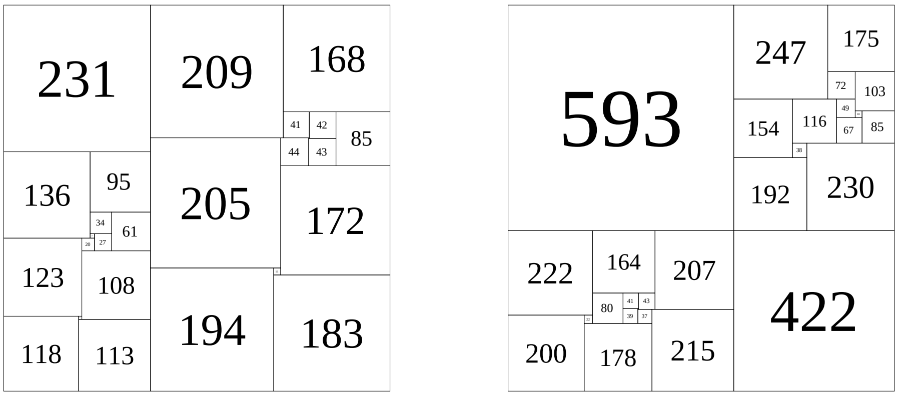
C.J. Bouwkamp published a series of papers [9, 11, 12, 10] in which he discussed methods for constructing squared rectangles and perfect squared squares. He gave a bouwkampcode listing[12] of the CPSS of order 39 with side 1813 discovered by Brooks, Smith, Tutte and Stone, but not shown in their 1940 paper.

1948

T.H.Willcocks, published[57] his discovery of a CPSS side 175, of order 24. It was constructed by overlapping two squared rectangles, one perfect and the other containing a single trivial imperfection involving a corner square. It held the record as the smallest known size and lowest order perfect squared square for the next thirty years, and was eventually found to be the CPSS of lowest possible order. See Figure 1 on page 2.

1950

W.T. Tutte published 'Squaring the Square'[52]. In this paper he described in more detail the general methods by which a square may be dissected into (smaller unequal non-overlapping) squares. Some new examples of such dissections were given. These included a CPSS of order 28 with side 1073. He also gave a CPSS of order 29 with side 1424, but the bouwkampcode is incorrect and most likely refers to a CPSS of order 29 with side 1399, later attributed to Federico.



26 : 608 W.T.T.   28 : 1015 A.H.S.

Figure 5: Tutte's 26:608 and Stone's 28:1015 (1940)

1951

T.H.Willcocks, published[58] his account of the methods he used to construct CPSSs of low order with small sizes. He included a number of new squared squares, these included his discovery of a new CPSS of order 26 with side 492, 4 new CPSSs of order 27 with sides 849, 867, 872 and 890, and a new CPSS of order 28 with side 577.

1963

P.J. Federico published[21] a paper in which he also provided a detailed account of CPSS construction methods. Federico gave a new general empirical method, by means of which 24 perfect squares of order below 29 were constructed. The CPSSs he gave in his paper included two new CPSSs of order 25, one with a side of 235 and the other with a side of 344, a new CPSS of order 26, with side 384, 7 new CPSSs of order 27 with sides 325, 408, 600, 618, 645, 648 and 825, then 11 new CPSSs of order 28 with sides 374, 714, 732, 741, 765, 765, 824, 1071, 1089, 1113 and 1137. He also gave 2 CPSS, both with a size of 1166, but the bouwkampcode was unconstructable. Federico defined the term *low order* [21, p.350] to mean the squared squares below order 29, he stated "this limit was chosen to avoid too long a list", and he also noted "twenty-nine perfect squares of order 29 were collected without attempting to apply fully the methods to this and higher orders". He only gave one example of a particular order 29 CPSS, size 468, indicative of the methods being illustrated in the list at the end of the paper. However in the paper itself he indicated how 7 new order 29 CPSSs were found, and gave sufficient information to work out their sizes, which were; 704, 724, 1341, 1377, 1412, 1457 and 1516.

1964

L'Udovit Vittek from Bratislava, Czech Republic also published[53] a CPSS of order 25 with size 235. This is the same order 25 size 235 published by Federico. Priority is given to Federico due to earlier publication.

In 1964 P. J. Federico[22] published a CPSS with side 429 of order 26 using a type of Fibonacci sequence construction published by S. Basin [7] in 1963.

1965-69?

E. Lainez, a Spanish engineer, constructed 2 CPSSs with sides 360, 460 of orders 26 and 27 respectively[28, p.67].

1972

In 1972 N.D. Kazarinoff and R. Weitzenkamp[**?**] used a graph theory analysis to limit the classes and specific cases of network that needed to be considered for graph generation and electrical network calculations on computer. In so doing they proved the non-existence of a CPSS of order less than 22.

1979

P. J. Federico published [23] "Squaring Rectangles and Squares, A Historical Review with Annotated Bibliography" in Proceedings of the Conference held in honour of Professor W.T. Tutte on the occasion of his sixtieth birthday. This was a comprehensive historical account of the problem of dividing a rectangle or squares into unequal squares. There was a detailed bibliography, extensively annotated by the author. The paper included all the latest developments, including Duijvestijn's 1978 discovery of the lowest order SPSS of order 21[20]. The paper also contained a number of tables. The table [23, p. 187] which we reproduce as Table 1 on page 8 contains counts of perfect squares in each order known in 1977. Compound 1 and Compound 2 refer to whether the compound perfect squared squares contain either 1 or 2 subrectangles.

Table 1: Number of Known Perfect Squares to Order 31 (1977)

Order	Simple	Compound 1	Compound 2
24	0	1	0
25	8	2	0
26	28	10	1
27	6	19	0
28	0	33	4
29	0	49	1
30	0	19	14
31	4	36	1

1979

P. Leeuw published his bachelor thesis[34] which proved that Willcocks 24:175

solution is the lowest order CPSS and the only CPSS of order 24. In his thesis [34, p.6] Leeuw stated "The idea of this way of solving the problem comes from P.J. Federico, the mapping into the computer, the development of the neccessary algorithms is performed by A.J.W. Duijvestijn and P. Leeuw." Leeuw's thesis was republished in a more expository form in the 1982 paper with Duijvestijn and Federico. This collaboration established the sought result; the Willcocks order 24 CPSS was produced and printed in the following manner[34, p.25];

8468469*11*111,94(0,81)(30,51)(64,31,29)(8,43)(2,35)(33)
8468468*13*111*94*(56,55)(16,39)(38,18)(3,4,9)(20,1)(5)(14)
The zero in the first line acted as a placeholder for the subrectangle in the second line.

The thesis methods produced 1883 CPSS, of these, 1942 were new, printed in the same manner as above but not shown in the paper.

1982

A.J.W. Duijvestijn, P.J. Federico, P. Leeuw published[3] their research into the lower limit of the order of compound perfect squared squares. This work was based on the 1979 thesis[34] by P. Leeuw, gave the same results with more extensive expository examples and some extra details on the CPSSs found.

Compound squares were considered separately in two types: Type 1, those that have only one subrectangle, and Type 2, those that have two subrectangles not having any element in common. The Type 2 did not produce any new CPSSs below order 30, so the work concentrated on Type 1. These were generated by using a modified electrical theory to transform squared rectangles into squared squares with one or more subrectangular inclusions. These are called deficient squares.

A deficient square is designated with a capital D and the number of squares in the deficient, which gives the order of the deficient. For example a Type 1 D15 is a deficient (squared) square with 15 squares and one subrectangle. A deficient with two subrectangles is designated with two D capitals (DD). If the included rectangle's aspect ratio could be matched to a perfect squared rectangle from known tables, then it could be scaled to fit in the deficient subrectangle. If a fit was found, and no two elements in the whole dissection were the same size, then a compound perfect squared square had been produced.

The task Duijvestijn, Federico and Leeuw set themselves was to find the lowest order CPSS. They achieved this by completely searching orders up to 24. They went beyond order 24 up to order 33, but they were not able provide definitive answers on orders higher than 24 as their squared rectangle tables only went to order 18. Their methods did however produce many higher order CPSSs. Duijvestijn, Federico and Leeuw stated they found 1942 new CPSSs, but only published two in their paper (26:483 and 28:816). The first table of their paper lists Type 1 results and the second table lists Type 2 results, and the third table lists the total number known by order at the time, with a breakdown of the 1942 newly discovered CPSSs by order. Listings of the bouwkampcodes of those 1942 CPSSs are not given. See also the same Type 1 and Type 2 totals in Leeuw's thesis[34, Page A-7]. On [3, page 25], the 1982 paper states that even within the scope of the program the results were possibly incomplete, "Numbers in italics are in those combinations of D's and rectangles that were not completely canvassed"; In the original table combinations of D's and rectangles from order 26 to 33 have been underlined, we take this as the reference to italics. In the reproduction of that table below, we have put the underlined entries in italics.

The first table (Table 1 in the 1982 paper) is reproduced here and referred to as Table 2;
From Table 2 it is clear that Willcock's order 24:175a CPSS had been found using

Table 2: Results for Type 1 Squares (1982)

D	24	25	26	27	28	29	30	31	32	33	Total
6	0										0
7	0	0									0
8	0	0	2								2
9	0	0	1	2							3
10	0	0	*2*	*1*	*4*						7
11	1	0	2	*1*	*2*	*15*					21
12	0	1	0	5	*3*	*8*	*42*				59
13	0	1	2	3	8	*13*	*32*	*86*			145
14	0	0	1	8	9	29	*46*	*131*	*214*		438
15	0	0	2	5	21	74	68	*91*	*294*	*768*	1323
Total (1)[a]	1	2	12	25	47	139	188	308	508	768	1998
Old (in)[b]	1	2	10	18	24	38	16	1	0	5	115
New [c]	0	0	2	7	23	101	172	307	519	763	1883
Old (out)[d]	0	0	0	1	9	11	3	38	8	50	120
Total (2)[e]	1	2	12	26	56	150	191	346	516	818	2118

(a) Results of program
(b) Squares in Total (1) already known
(c) Difference
(d) Known squares outside scope of program
(e) Total squares now known

this process. The two order 25 CPSSs found by Federico in 1962 (25:235a and 25:344a) were also found, but as perfect squared rectangles (PSR) for order 19 were not available at the time, it was possible that a D6 might combine with one or more order 19 PSRs or a D16 might combine with an order 9 SPSR to produce more order 25 CPSSs. We now know[5] that this is not possible, and order 25 CPSSs were completed in 1962 by Federico.

Table 2 shows Order 26 has two new discoveries, one of them, 26:483a, is shown in the paper[3, p.25]. The other is not shown. There were 10 Type 1 CPSSs of order 26 known at the time, these were found by Federico (4), Willcocks (1), Bouwkamp (4), and Lainez (1). Federico and Willcocks had already published their discoveries, (except for 26:638a by Federico). Bouwkamp's CPSSs all featured deficients of low order (D8, D9, D10), and are undated and unpublished, we assume they were constructed by hand prior to 1977. If we classify Bouwkamp's 4 CPSSs by which deficient order they belong to, they are fully accounted for in Table 2. Only one Type 2 exists in order 26 CPSS, Tutte's 26:608a, and it was found. CPSSs constructed from D6 and D16 were not in the scope of Leeuw's program and were not produced until years later by Skinner, when as we now know, he completed the process of discovery in order 26 CPSSs, finding two D6 CPSSs (26:480a, 26:648a) and one D16 CPSS (26:493a).

Table 3: Type 1 and 2 Results for Orders 24 -29 in 1982 and 2010, 2012

Order		24		25		26		27		28		29
D / Year	1982	2010	'82	'10	'82	'10	'82	'10	'82	'10	'82	'12
D6	0	0	-	0	-	2	-	3	-	12	-	22
D7	0	0	0	0	-	0	-	0	-	2	-	3
D8	0	0	0	0	2	2	-	0	-	11	-	24
D9	0	0	0	0	1	1	2	2	-	4	-	11
D10	0	0	0	0	2	2	1	1	4	6	-	22
D11	1	1	1	1	2	2	1	1	2	3	15	18
D12	0	0	1	1	0	0	5	5	3	5	8	10
D13	0	0	0	0	2	2	3	3	8	7	13	16
D14	0	0	0	0	1	1	8	8	9	8	29	29
D15	0	0	0	0	2	2	5	5	21	21	74	70
D16	-	-	-	0	-	1	-	8	-	23	-	36
D17	-	-	-	-	-	0	-	6	-	15	-	35
D18	-	-	-	-	-	-	-	4	-	12	-	30
D19	-	-	-	-	-	-	-	-	-	9	-	37
D20	-	-	-	-	-	-	-	-	-	-	-	46
In scope	1	1	2	2	12	12	25	25	47	50	139	143
Type 2	0	0	0	0	1	1	0	0	4	5	0	3
Total 1 & 2	1	1	-	2	-	16	-	46	-	143	-	412

In Table 3 we compare CPSSs of Type 1 found in 1982 to those found in 2010 and 2012, according to the number of deficient squares found in each order 24 to 29. In the 1982 results the range of deficient squares narrows as the order increases because the orders of squared rectangles needed for substitution into, and generation of deficient squares, increases as the order of the CPSS increases, and squared rectangle catalogues did not exist past order 18 at the time.

We have not attempted an analysis of the 1982 paper results on orders 30, 31, 32 and 33 because these orders are still incompletely enumerated.

Using Table 3 on page 11 we can compare deficient square totals of CPSSs of order 26, in the D8 to D15 range, from 1982 to those of 2010. We find there are the same number, 12 of them, and the numbers in each deficient square order match exactly.

This leaves just the other unpublished discovery of the 1982 paper to be accounted for in order 26. The only remaining position for it in the table is for another D15. We now know it is 26:512a, (See Figure 7 on page 16). It was also found by Ian Gambini in 1999, given implicitly in an isomer count [26, p.25, Tab. 2.6], but he did not identify it. It remained generally unknown until rediscovered for the third time by Anderson and Pegg in 2010[5].

The 1982 paper's results for order 27 featured 25 Type 1 CPSSs in program scope from D9 to D15. Of these 7 were reported as new discoveries. If we compare these to the CPSSs of order 27 which were enumerated in 2010[5], there are also 25 of them in the D9 to D15 range, and the numbers in each deficient order match exactly the numbers given in 1982. So after eliminating the 18 known Type 1 CPSSs in program range, we can deduce that the 7 remaining new discoveries were 27:599a, 27:636a, 27:861a (rediscovered by Skinner), 27:804a, 27:820a, 27:824a (rediscovered by Anderson and Pegg 2010) and 27:931a (rediscovered by Morley). There are no Type 2 CPSS in order 27.

The CPSSs of order 28 were completely enumerated in 2010 and order 29 in 2012[5]. If we compare the Type 1 results in orders 28 and 29 from 2010 and 1012 with the Type 1 results listed in 1982 (our Table 3), we have difficulty identifying which CPSSs were discovered as the deficient totals mostly do not match.

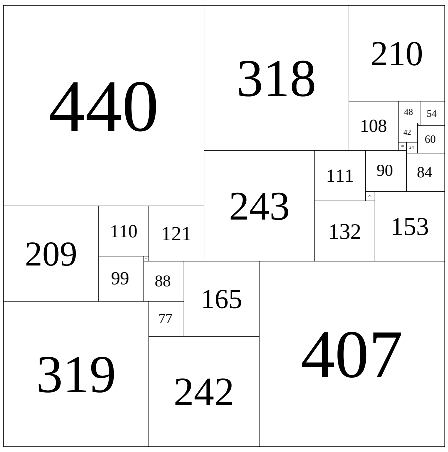
The Type 1 program counts from 1982 are not accurate enough to be able to deduce which particular CPSSs of order 28 and 29 were discovered in 1979. We are unable to put the deficient squares (within the scope of the 1979 program) into one-to-one correspondence with the CPSSs that are now known to exist in those orders. If the 1979 bouwkampcode Type 1 printout listings were still available we would be able to determine which discoveries had been made.

The paper also looked at Type 2 CPSSs and stated [3, p.27] "The results ... showed that there were no type 2 squares of order 24 or lower. This field had already been pretty well worked over, and no new squares below order 30 were found." However an additional Type 2 CPSS, 28:471a has recently been found in order 28 and three new Type 2 CPSSs were found in 2011 in order 29[5]. Of these, 28:471a, 29:569a and 29:966a should have been found by the methods in the 1982 paper but were not.

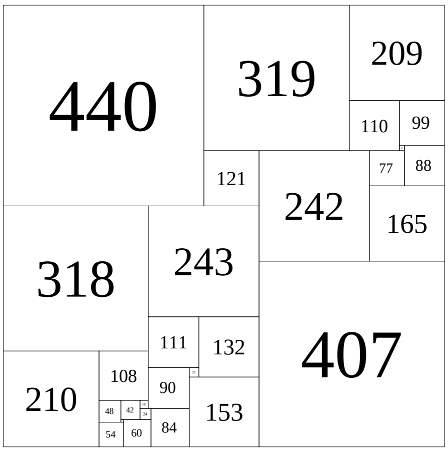
The aim of the paper was not enumeration of CPSSs across a range of orders, but rather to establish the lowest order for CPSSs. This objective was achieved.

1990

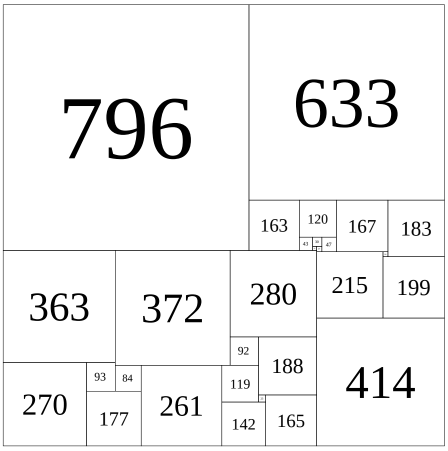
J. D. Skinner produced 2 new CPSSs of order 29 using a technique of T. H. Willcocks



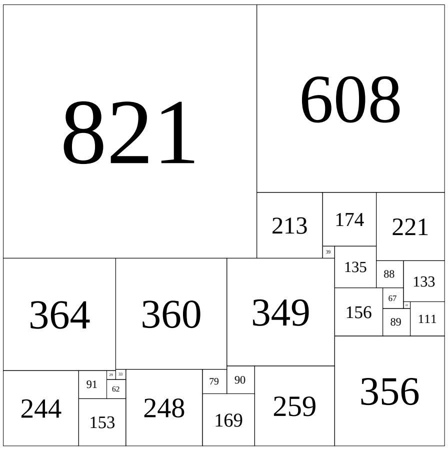
29 : 968c CJB (1967)



29 : 968d CJB (1967)



29 : 1429a JDS (1990)



29 : 1429b JDS (1990)

Figure 6: Bouwkamp's 29:968 CPSS pair (elements pairwise the same) and Skinner's 29:1429 CPSS pair (elements pairwise different)

(1951) Technique 2.211 [58, p.305] applied to two compound perfect squares of 28th-order: the first of reduced side 1015 due to A. H. Stone (1940) and the second of reduced side 1073 described by W. T. Tutte (1950)[52]. The result was a pair of 29th-order compound perfect squares of reduced side 1429 with no common element [44]. Around this time Skinner found 26:480a , 26:493a and 26:648a which completed the discovery process in order 26 CPSS.

1991

C. J. Bouwkamp published 'On some new simple perfect squared squares' [13] which featured new low order SPSSs of order 24 and 25. This paper also featured two CPSSs of order 29 discovered by Bouwkamp in 1967 but previously unpublished. These CPSSs have the same side of 968 and the same elements arranged differently. These two CPSSs are isomers, but unlike most CPSS isomers where the included rectangle is arranged differently, in this case it is the elements surrounding the included rectangle that are arranged differently. They are counted as separate CPSSs.

See Figure 6 on page 13 for images of Skinner's and Bouwkamp's CPSS pairs.

1999

Ian Gambini published his doctoral thesis *Quant aux carrés carrelés* on squared squares[26]. He used several different methods to enumerate perfect squared rectangles and squares.

He implemented his version of what he called the *classical method*. That is, he generated non-isomorphic 2-connected planar graphs (with minimum degree 3 to ensure perfect dissections) and solved the Kirchhoff equations for electrical networks of the graphs to find the sizes of the squares in the dissection corresponding to edges with unit resistences. His graph generation method, unlike Duijvestijn's, did not use Tutte's wheel theorem[32]. Gambini was able to generate graphs with up to 25 edges and produce simple and compound perfect squared rectangles (SPSRs and CPSRs) to order 24. Within these solutions he found the known CPSSs and simple perfect squared squares (SPSSs) up to and including order 24. He published a table of SPSR and CPSR counts up to and including order 24.

Gambini observed that a perfect squared square can only have one side with a minimum of 2 squares along an edge. Hence only one of the polar vertices in the graph, or its dual, can have a vertex of degree 3. He thereby constrained the graph generation algorithm and eliminated some graphs from production which could not produce squared squares. Gambini continued the 'classical method' beyond order 24 for perfect squared squares and produced all to order 26. Table 2.6 of Gambini's thesis listed SPSS and CPSS isomers counts up to and including order 26. In the SPSS counts Gambini obtained the same results as Duijvestijn. In the CPSS counts Gambini identified;

- 4 isomers of order 24 CPSS

- 12 isomers in order 25 CPSSs
- 100 isomers in order 26 CPSSs.

Gambini did not associate the isomers with particular CPSSs , however we can match them up with known discoveries of that time.

- The 4 isomer counts in order 24 corresponded to T.H. Willcock's 24:175a CPSS (4 isomers).
- The 12 isomer counts in order 25 corresponded to P.J. Federico's 25:235a (4 isomers) and 25:344a (8 isomers).
- The 100 isomer counts of order 26 corresponded to a total of 92 isomers derived from 15 known order 26 CPSSs (isomer counts in parentheses); 288a(4), 360a(4), 360b(4), 384(4), 429a(4), 440a(4), 480a(4), 483a(4), 492a(4), 493a(4), 500a(16), 608a(16), 612a(4), 638a(8), 648a(4) and an additional 8 isomers not associated with any CPSS(s) known at the time.

The 8 isomer discrepency was not resolved until 2010. The additional CPSS which completed the order has a side of 512 and has 8 isomers. This CPSS was deduced to have been discovered by Duijvestijn, Federico & Leeuw in 1979 but not published and finally identified until 2010 by Anderson and Pegg. This CPSS completes the catalogue of order 28. Please see Figure 7 on page 16 for an illustration of CPSS 26:512a.

Gambini also developed new methods of producing perfect squared squares using several tiling algorithms. He improved the efficiency of his algorithms by proof of theoretical bounds he established on the minimum sizes possible for elements on both the boundary sides (size of 5) and corners (size of 9) of a perfect squared square. He was able to produce a large number of SPSS across an unbroken range of orders from order 21 to order 128. He proved that the 3 SPSS with sides of 110, originally found by Duijvestijn and Willcocks, are the minimum possible size for a perfect squared square. He produced only one new CPSS (of order 52, side 976).

Using a variation on his tiling algorithm Gambini was also able to find perfect squared cylinders and a perfect squared torus (of order 24 with side 181).

2010

Richard K. Guy, Ed Pegg Jr and Stuart Anderson collaborated to extend the known solutions to the Mrs Perkin's quilt problem[16, 50, 31, 6]. Mrs Perkin's quilts include all combinations of simple, compound, perfect and imperfect squared squares. Using Brendan McKay and Gunnar Brinkmann's planar graph generation software *plantri* [15, 14] and electrical network tiling software written with `C++` standard libraries and Boost Ublas library (by Anderson), Anderson and Pegg enumerated all perfect squared squares and simple imperfect squared squares (SISSs) to order 28 [5]. As a subset of the quilt enumeration Anderson and Pegg produced all CPSSs up to and including order 28. The CPSS counts by order are;

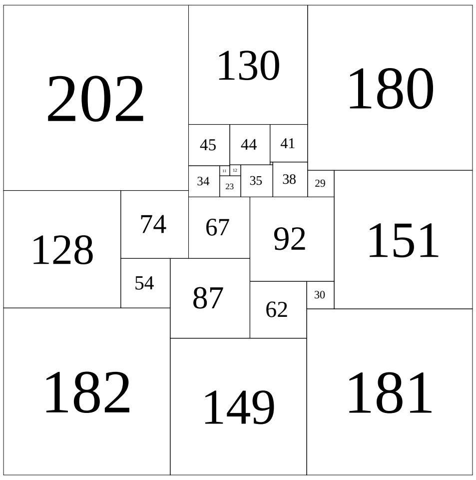


Figure 7: Duijvestijn, Federico & Leeuw 26:512a (1979)

- 1 CPSS of order 24, with 4 isomers (Willcocks, 1948)
- 2 CPSSs of order 25, with 12 isomers (Federico, 1962)
- 16 CPSSs of order 26, with 100 isomers, including 1 CPSS, 8 isomers, with side 512 not previously identified, (discovered in 1979 by Duijvestijn, Federico and Leeuw and rediscovered by Ian Gambini in 1999) which completed this order.
- 46 CPSSs of order 27, with 220 isomers, including 4 CPSSs not previously known (with sides 345a, 624a, 648b, 857a), and 3 CPSSs which had been discovered by Duijvestijn, Federico and Leeuw in 1979, but never published, (27:804a, 27:820a and 27:824a) which completed this order.
- 143 CPSSs of order 28, with 948 isomers, including 50 CPSSs not previously known, which completed this order. Duijvestijn, Federico and Leeuw found 23 new CPSSs in this order but we do not know which ones they found.

2011

S.E. Anderson and Stephen Johnson commenced enumeration of order 29 CPSSs, and processed all 2-connected minimum degree 3 graphs with up to 15 vertices. That left the largest graph class, the 16 vertex class, still to be processed.

2012 January-October

In March 2012 G.H. Morley used SPSR substitution into existing CPSSs to discover more new CPSSs in order 29 and in the order thirties[39]. S.E. Anderson used computer substitution of squared squares into squared squares to discover large numbers (millions!) of CPSS's in orders 40s and 50s[5].

2012 October-November

S.E. Anderson rewrote his software and over a 9 day period, processed the remaining 16 vertex, 2-connected, minimum degree 3, 30 edge graphs using 34 processor cores on the Amazon Elastic Cloud supercomputer. Combined with the earlier 13, 14 and 15 vertex, 30 edge 2-connected graphs processed by Pegg, Johnson and Anderson, and Morley's recent discoveries in order 29, this completed the enumeration of order 29 CPSSs. The final count for order 29 CPSSs is; 412 CPSSs of order 29, with 2308 isomers, including 253 CPSSs not previously known, which completed this order[5]. Duijvestijn, Federico and Leeuw found 101 new CPSSs in order 29, but we do not know which ones they found.

2013 January

James B. Williams [59] wrote a square tiling program to search for perfect squared squares in the order twenties and thirties. He found no new CPSS in the order twenties but his discoveries in the order thirties were mostly new;

- 1064 order 30 CPSS isomers
- 2959 order 31 CPSS isomers
- 7605 order 32 CPSS isomers
- 19612 order 33 CPSS isomers

3 Theory and Computer methods

3.1 Dissection, tiling and the trivial dissection

The dissection of a rectangle into squares can be viewed as a tiling by squares of a rectangle. A square packing of a rectangle is a set of squares embedded in the rectangle with no overlaps. A square covering of a rectangle is a set of squares embedded in the rectangle with no gaps. A square tiling is both a square covering and a square packing, à la Grunbaum and Shephard [27, p16.].

In the case where a rectangle is a square, it can be tiled with a single square. This differs from the dissection of a square into squares. In the dissection of a square into squares such a thing is called the 'trivial' dissection. It has been the convention in the history of squared squares that the trivial dissection is not included as a solution to the problem of squaring the square. There are probably several reasons for this;

- excluding the trivial dissection as a solution, changes the problem from one with a trivial solution to one that is difficult and challenging.
- including the trivial dissection as a solution, would provide a trivial squared tiling solution to every integer sized square. These infinite trivial solutions would be

essentially the same as the trivial solution of the unit square and provide us with no new information.

- dissection evokes the physical operation of marking or cutting into a surface with horizontal and vertical lines. The trivial dissection requires no such cuts or marks and in that sense is not a dissection at all.
- allowing the trivial dissection would require it to be included as an extra solution for each non-trivial solution found. We could say Willcocks's order 24 perfect squared square is really an order 25 dissection, and similarly for all other perfect squared squares. This is superfluous.

3.2 Graph theory terms

We introduce some informal graph theory terminology we will be using in the next section of the paper.

Graphs are mathematical objects. They consist of **vertices** (or nodes) and **edges** (which connect the vertices).

An **undirected graph** is one in which edges have no orientation. In an undirected graph the pair of vertices in a edge is unordered, $(v_0, v_1) = (v_1, v_0)$ and a **directed graph** is one in which each edge is a directed pair of vertices, $(v_0, v_1) \neq (v_1, v_0)$.

If (v_0, v_1) is an edge in an undirected graph, v_0 and v_1 are **adjacent**. The edge (v_0, v_1) is **incident** on vertices v_0 and v_1.

If (v_0, v_1) is an edge in a directed graph, v_0 is **adjacent to** v_1, and v_1 is **adjacent from** v_0. The edge (v_0, v_1) is **incident** on v_0 and v_1.

The **degree** of a vertex is the number of edges incident on that vertex. In directed graphs, the **in-degree** of a vertex v is the number of edges that have v as the head and the **out-degree** of a vertex v is the number of edges that have v as the tail.

A **weighted graph** is a graph with numbers (weights) associated with each edge.

A **simple** graph is an undirected graph containing no loops or multiple edges. An edge which connects a vertex to itself is a **loop**.

A **path** is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that consecutive vertices v_i, and v_{i+1} are adjacent. A simple path is one with no repeated vertices and a **cycle** is a simple path except the last vertex is the same as the first vertex.

A **connected graph** is a graph where any two vertices are connected by some path.

A graph is called **k-connected** if one must remove *at least* k vertices (and the edges adjacent to those vertices) in order to separate the graph into disconnected parts. If there is some set of k vertices that, when removed, achieves the separation, we say the graph is **exactly k-connected**.

A **planar** graph is a graph that can be *embedded* in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. Every planar graph can be drawn on the sphere and vice versa.

A **planar map** is the combinatorial embedding of a planar graph, i.e., the map M is bidirected (for every edge v_0 and v_1 of M the reverse edge v_1 and v_0 is also in M) and there is a planar embedding of M such that for every node v the ordering of the edges in the adjacency list of v corresponds to the counter-clockwise ordering of these edges around v in the embedding.

Isomorphic graphs are graphs which contain the same number of graph vertices connected in the same way.

A **subgraph** is a subset of vertices and edges forming a graph. A **connected component** is a maximal connected subgraph.

A **tree** is a connected graph with no cycle.

A **spanning tree** of a graph G is a subgraph of G which is a tree that includes all the vertices of G.

3.3 The p-nets and c-nets of a squared rectangle

We associate a network graph with a squared rectangle such that each horizontal line segment of the squared rectangle corresponds to a graph node and each square corresponds to a graph edge (or branch in electrical terminology) connecting the two nodes of the top and bottom horizontal lines of the square. We put an arrow on each branch to indicate the positive direction for currents running through the graph. The nodes $P(+)$ and $P(-)$ are the poles of the network.

There is another p-net (and c-net) we can associate with the squared rectangle, this is the dual graph and corresponds to the same construction applied instead to the vertical line segments of the squared rectangle.

In each of the branches of either the network graph, or it's dual graph, a unit resistance is placed. Electricity acts according to the equation $V = IR$ where V is the voltage drop and I the current and R the resistance. If we assume the resistance is 1, then the current is the voltage drop. Thus if we think of an branch $i \to j$ as a wire having resistance 1 with voltage v_i at i and voltage v_j at j, then the current from i to j is the voltage drop $v_i - v_j$.

The current in each branch is given in terms of a current variable C called the *Complexity*, entering at $P(+)$, and leaving at $P(-)$. Kirchhoff's current law then gives n equations for the branches incident on n nodes in n unknown potentials, but one equation is redundant and can be eliminated and we can also set the potential at $P(-)$ to 0 and hence remove this node voltage variable. This gives $n-1$ independent linear equations and $n-1$ unknowns so a unique solution to the equations is always possible.

The value of C is then reduced so as to make the currents all integers without any common factor. These are the 'reduced' currents, the numbers attached to the branches which are also the side lengths of the component squares.

The network graph superimposed on the squared rectangle called a p-net (polar net). If the two nodes $P(+)$ and $P(-)$ are connected by a new branch the net is completed and is called a c-net (completed net). The c-nets are planar 3-connected planar graphs, this means one must remove at least 3 nodes (and the branches adjacent to those nodes) in order to separate the c-net into disconnected parts. By a result of Steinitz planar 3-connected planar graphs are isomorphic to edge skeleta of polyhedra.

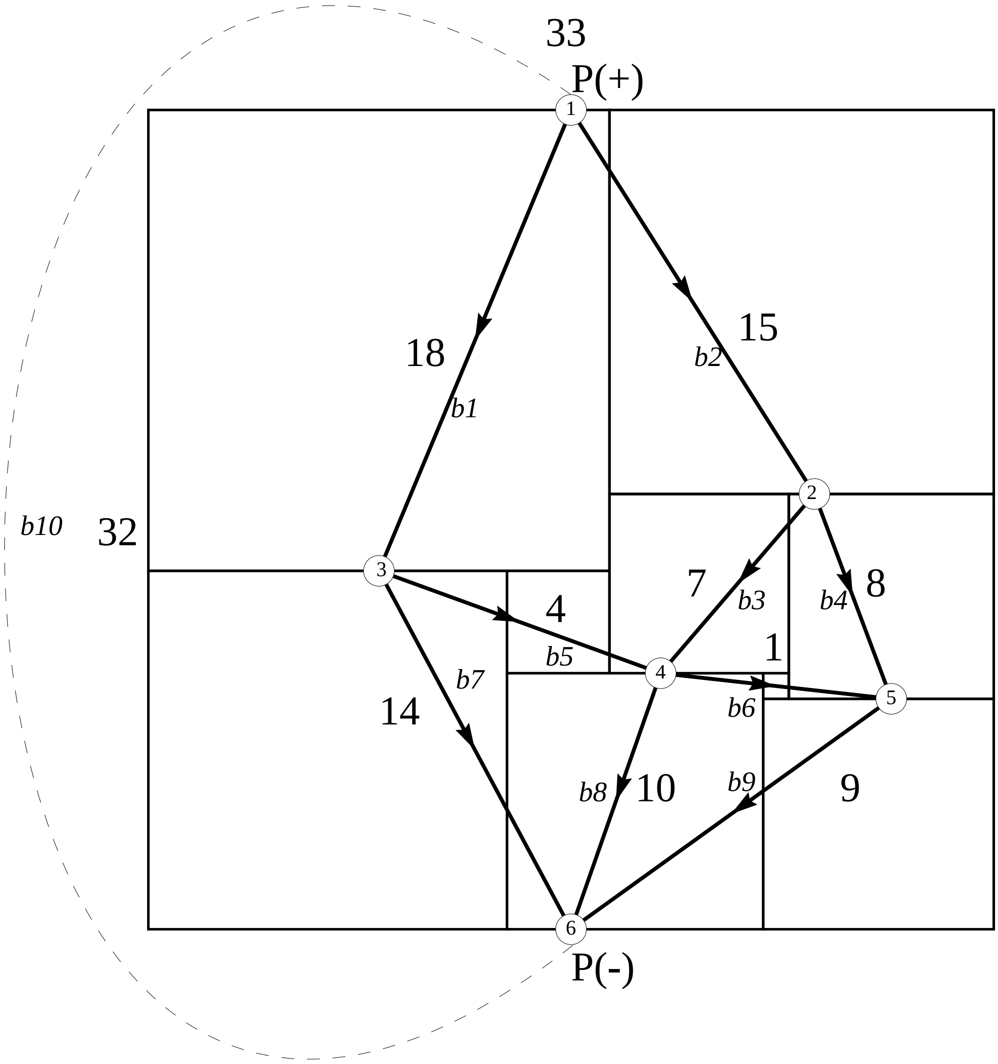


Figure 8: 33 x 32 simple perfect squared rectangle and p-net

It was proved in the 1940 Brooks, Smith, Tutte, Stone paper that every simple squared rectangle can be derived from a c-net. If the c-net has m edges, m p-nets are produced by removing each edge in turn, and hence at least m squared rectangles of order $m-1$ are obtained. The process is equivalent to placing a battery in turn in each edge of the c-net and calculating the relative values of the currents in the other edges.

Not every squared rectangle produced in this manner will be necessarily perfect, but every simple perfect rectangle of order $m-1$ is produced from the complete set of c-nets of order m.

3.4 Electrical network definitions

We also introduce some electrical engineering terminology and a matrix definition to be able to calculate electrical voltages and currents in a given networks, and thereby show how squared rectangle dissections can be produced;

An **electrical network** is an interconnection of electrical elements.

An **electrical circuit** is a network consisting of a closed loop, giving a return path for the current.

A **resistive circuit** is a circuit containing only resistors and ideal current and voltage sources. For a network composed of linear components, such as a resistive circuit there will always be one, and only one, unique solution for a given set of boundary conditions.

Network analysis is the process of finding the voltages across, and the currents through, every component in the network.

Kirchhoff's Current Law (KCL) For any electrical circuit, for any of its nodes, the algebraic sum of all branch currents leaving the node is zero.

Kirchhoff's Voltage Law (KVL) For any electrical circuit, for any of its loops, the algebraic sum of all branch voltages around the loop is zero.

Incidence Matrix

The (branch-node) incidence matrix of a graph is defined as follows;

$$A_{ik} = \begin{cases} 1 & \text{if branch } k \text{ is directed away from node } i \\ -1 & \text{if branch } k \text{ is directed towards node } i \\ 0 & \text{if branch } k \text{ is not incident on node } i \end{cases}$$

The direction of a branch is the *reference direction*, this can be an arbitrary choice, but applied consistently to the network. For example, each node of the network graph is indexed with an integer, if the direction of an edge is from a lower node index to higher node index, we can say it is directed away from the lower node and give a value of 1 in the branch-node incidence matrix. Alternatively if the edge is going from a higher node index to a lower node index we can say it is directed towards the lower node and give a value of -1 in the branch-node incidence matrix.

3.5 An example of the calculation of squared rectangles from a planar graph

Applying the definition of incidence matrix to the network graph of Figure 8 on page 20 we obtain the branch-node incidence matrix A_a ;

$$A_a = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{pmatrix}$$

We form a vector j where j_k is the current in branch b_k. The equation $Aj = 0$ gives Kirchhoff's Current Law (KCL).

$$A_a j = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \\ j_9 \\ j_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

From inspection of the network graph of Figure 8 on page 20 it is clear the rows of $A_a j$ give the branch current equations of KCL at each node.

$$= \begin{pmatrix} j_1 & j_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -j_{10} \\ 0 & -j_2 & j_3 & j_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -j_1 & 0 & 0 & 0 & j_5 & 0 & j_7 & 0 & 0 & 0 \\ 0 & 0 & -j_3 & 0 & -j_5 & j_6 & 0 & j_8 & 0 & 0 \\ 0 & 0 & 0 & -j_4 & 0 & -j_6 & 0 & 0 & j_9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -j_7 & -j_8 & -j_9 & j_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

If we add at the KCL equations (written in terms of the branch currents $j_1, j_2, ..., j_{10}$), all the 6 KCL equations cancel out. Since every branch must leave one node and terminate on another node, all branch currents will cancel out in the sum of the 6 equations. We conclude that *the 6 equations obtained by writing KCL for each of the nodes of the network graph are linearly dependent.*

Now we pick a node, the polar node $P(-)$, called the *datum node*, and form another incidence matrix , including all nodes *except* $P(-)$, we call this the *reduced* incidence matrix A.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Clearly A is the same matrix as A_a except one row (the last) has been removed. With A we can apply $Aj = 0$ (KCL) and form 5 equations. By removing one of the equations it can always be shown that the remaining equations are *linearly independent*.

$$Aj = \begin{pmatrix} j_1 & j_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -j_{10} \\ 0 & -j_2 & j_3 & j_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -j_1 & 0 & 0 & 0 & j_5 & 0 & j_7 & 0 & 0 & 0 \\ 0 & 0 & -j_3 & 0 & -j_5 & j_6 & 0 & j_8 & 0 & 0 \\ 0 & 0 & 0 & -j_4 & 0 & -j_6 & 0 & 0 & j_9 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

To obtain equations for potential differences in the graph we use the transpose of A. The transpose of A is obtained by replacing all elements A_{ik} with A_{ki}. In other words, the matrix transpose, most commonly written A^T, is the matrix obtained by exchanging A's rows and columns.

Kirchhoff's Voltage Law (KVL) states that for a network, for any loop, the sum of the potentials (voltage drops) around the loop is zero. (A loop is a subgraph of the network which is connected and has exactly 2 branches of the subgraph incident with each node).

We form an equation for KVL, $v = A^T e$ where the components e_i of the vector e describe the electrical potential at the nodes i of the graph, and v is a vector describing the *difference* in potential across each branch k of the graph. We apply KVL to the network graph of Figure 8 on page 20 to obtain the branch voltages from the node voltages.

$$v = A^T e = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix}$$

Inspection of the network graph of Figure 8 on page 20 demonstrates v_k corresponds to the voltage drop in each branch b_k.

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \end{pmatrix} = \begin{pmatrix} e_1 & 0 & -e_3 & 0 & 0 \\ e_1 & -e_2 & 0 & 0 & 0 \\ 0 & e_2 & 0 & -e_4 & 0 \\ 0 & e_2 & 0 & 0 & -e_5 \\ 0 & 0 & e_3 & -e_4 & 0 \\ 0 & 0 & 0 & e_4 & -e_5 \\ 0 & 0 & e_3 & 0 & 0 \\ 0 & 0 & 0 & e_4 & 0 \\ 0 & 0 & 0 & 0 & e_5 \\ -e_1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So far we have formed a reduced incidence matrix from the network graph and have derived the Kirchhoff equations of KCL and KVL. We can combine these matrix equations by starting with Ohm's Law and using substitution;

$$\begin{aligned}
v &= jr && \text{by Ohm's Law,} && (1)\\
j &= (1/r)v && \text{rearranging} && (2)\\
j &= Gv && \text{conductance matrix } G = 1/r, && (3)\\
Aj &= AGv && \text{premultiply by A,} && (4)\\
Aj &= AG(A^T e) && \text{(KVL) } v = A^T e\ , && (5)\\
AG(A^T e) &= 0 && \text{(KCL) } Aj = 0\ \text{, swap } lhs, rhs && (6)\\
AI(A^T e) &= 0 && G = I\text{, all conductances are 1} && (7)\\
(AA^T)e &= 0 && AI = A,\ G \text{ is the identity matrix } I, && (8)\\
Ke &= 0 && \text{define } AA^T = K\text{; the \textit{Kirchhoff} matrix} && (9)
\end{aligned}$$

We continue with the example from the network graph of Figure 8 on page 20 to obtain it's Kirchhoff matrix.

$$K = AA^T = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Kirchhoff matrix of Figure 8. The last row of the matrix, giving equations for branches connected to P(-) has been eliminated. This node, the negative pole is also called the ground, or reference or datum node.

$$K = AA^T = \begin{pmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 3 & -1 & 0 \\ 0 & -1 & -1 & 4 & -1 \\ 0 & -1 & 0 & -1 & 3 \end{pmatrix}$$

We can invert the square matrix K to solve for e, then substitute e into KVL to obtain v, which also gives j, ($j = v$ as all conductances are 1). We interpret the values of e as the horizontal dissection lines in the squared rectangle and the branch currents j as the dissected square sizes.

However we have not specified any source currents or voltages so all values are relative not absolute. We can remedy this by calculating a number based on the network graph, we call this number the Complexity, it is the determinant of the Kirchhoff matrix, and gives the number of spanning trees of the graph; $\tau(G)$. This becomes the total current entering at the positive pole and leaving at the negative pole. We multiply the inverted Kirchhoff matrix K by the Complexity, $\tau(G)$, to get another matrix V from which we obtain integer values for node voltages e and from these the branch currents. These are known as the 'full' voltage and currents, these can often be reduced by a common factor.

$$\det(K) = \tau(G) \qquad \text{the number of spanning trees of } G \tag{10}$$

$$\det(K)K^{-1}e = V \qquad V \text{ gives the 'full' node voltages} \tag{11}$$

In the example of Figure 8 the determinant of K is 130, which is also the number of spanning trees of the graph. We then calculate V for Figure 8.

$$\det(K)K^{-1} = V = \begin{pmatrix} 64 & 34 & 28 & 20 & 18 \\ 34 & 79 & 23 & 35 & 38 \\ 28 & 23 & 61 & 25 & 16 \\ 20 & 35 & 25 & 55 & 30 \\ 18 & 38 & 16 & 30 & 66 \end{pmatrix}$$

Each indexed row and column have the same entries, which are sets of node voltage solutions satisfying the Kirchhoff equations. To enumerate squared rectangles we need to find all branch currents solutions of the network graph. To do this we form a triple matrix product, sandwiching $V = \det(K)K^{-1}$ between A^T and A to obtain a 'full' currents matrix F with solutions where each branch, in turn, acts as the polar edge ;

$$F = A^TVA \qquad \text{triple matrix product gives 'full' currents matrix } F \tag{12}$$

$$\tag{13}$$

$$F = \begin{pmatrix} 69 & 25 & 16 & 9 & -28 & -7 & -33 & -5 & 2 \\ 25 & 75 & -30 & -25 & 20 & 5 & 5 & -15 & -20 \\ 16 & -30 & 64 & 36 & 18 & -28 & -2 & -20 & 8 \\ 9 & -25 & 36 & 69 & 2 & 33 & 7 & 5 & -28 \\ -28 & 20 & 18 & 2 & 66 & -16 & 36 & -30 & -14 \\ -7 & 5 & -28 & 33 & -16 & 61 & 9 & 25 & -36 \\ -33 & 5 & -2 & 7 & 36 & 9 & 61 & 25 & 16 \\ -5 & -15 & -20 & 5 & -30 & 25 & 25 & 55 & 30 \\ 2 & -20 & 8 & -28 & -14 & -36 & 16 & 30 & 66 \end{pmatrix}$$

We need to obtain the 'reduced' currents from the 'full' currents. To do this we form a vector R, the reduction vector, composed of the GCD (greatest common divisor) of each row of the full currents matrix F, then divide F by R to obtain the reduced currents matrix B.

$$\vec{R}_i = \operatorname*{gcd}_{j=1}^{m} F_{ij} \qquad \text{GCD to rows of } F \text{ obtains the reduction vector } R \tag{14}$$

$$F/R = B \qquad \text{dividing F by R gives the reduced currents matrix } B \tag{15}$$

$$F/R = \begin{pmatrix} 69 & 25 & 16 & 9 & -28 & -7 & -33 & -5 & 2 \\ 25 & 75 & -30 & -25 & 20 & 5 & 5 & -15 & -20 \\ 16 & -30 & 64 & 36 & 18 & -28 & -2 & -20 & 8 \\ 9 & -25 & 36 & 69 & 2 & 33 & 7 & 5 & -28 \\ -28 & 20 & 18 & 2 & 66 & -16 & 36 & -30 & -14 \\ -7 & 5 & -28 & 33 & -16 & 61 & 9 & 25 & -36 \\ -33 & 5 & -2 & 7 & 36 & 9 & 61 & 25 & 16 \\ -5 & -15 & -20 & 5 & -30 & 25 & 25 & 55 & 30 \\ 2 & -20 & 8 & -28 & -14 & -36 & 16 & 30 & 66 \end{pmatrix} / \begin{pmatrix} 1 \\ 5 \\ 2 \\ 1 \\ 2 \\ 1 \\ 1 \\ 5 \\ 2 \end{pmatrix}$$

$$= B = \begin{pmatrix} \mathbf{69} & 25 & 16 & 9 & -28 & -7 & -33 & -5 & 2 \\ 5 & \mathbf{15} & -6 & -5 & 4 & 1 & 1 & -3 & -4 \\ 8 & -15 & \mathbf{32} & 18 & 9 & -14 & -1 & -10 & 4 \\ 9 & -25 & 36 & \mathbf{69} & 2 & 33 & 7 & 5 & -28 \\ -14 & 10 & 9 & 1 & \mathbf{33} & -8 & 18 & -15 & -7 \\ -7 & 5 & -28 & 33 & -16 & \mathbf{61} & 9 & 25 & -36 \\ -33 & 5 & -2 & 7 & 36 & 9 & \mathbf{61} & 25 & 16 \\ -1 & -3 & -4 & 1 & -6 & 5 & 5 & \mathbf{11} & 6 \\ 1 & -10 & 4 & -14 & -7 & -18 & 8 & 15 & \mathbf{33} \end{pmatrix}$$

Each row of B corresponds to a set of square sizes in a squared rectangle. B is indexed by the edges of the network graph. B is a square matrix and the diagonal entries correspond to the (reduced) current in the polar edges, that is, the width of each squared rectangle solution. In the theory of squared rectangles, the semiperimeter of the rectangle is equal to $\det(K)$. The height can then be calculated as the diagonal entry B_{ii} (width) subtracted from $\det(K)/R_i$. Width may be less than height at this stage, a standard orientation is imposed later.

A number of the entries in B are negative. The negative values correspond to current directions along edges which are a reversal of the original reference directions. To change the negative values to positive currents we reverse the reference directions of those edges in the network graph.

Among the squared rectangle solutions for the Figure 8 graph found in B are 3 unique squared rectangles of order 9. There are 2 simple perfect squared rectangles (33x32 and 69x61) (see Figure 3 on page 4) and 1 simple imperfect squared rectangle (15x11).

3.6 Squared squares

In the case where the height is equal to the width, the squared rectangle is a squared square, and if no two squares are the same size, it is a perfect squared square. In the matrix B, if any diagonal entry $B_{ii} = \det(K)/2R_i$ then a squared square of reduced size B_{ii} has been found.

3.7 Bouwkampcode; encoding the dissections

Since Bouwkamp, squared rectangles have often been represented using a code (called bouwkampcode). "First we suppose the rectangle to be drawn out in such a manner that its largest sides are horizontal. Then the element in the upper left corner should not be smaller than the three remaining corner elements. .... Henceforth we will always "orient" a squared rectangle in the above sense ... . Now the given oriented rectangle is squared by horizontal and vertical line segments. Consider the group of elements with their upper horizontal sides in a common horizontal segment. The individual elements of this group are conveniently ordered by a reading from left to right. The various groups themselves are ordered according to upwards downwards reading, starting with the upper horizontal side of the given rectangle. If necessary line segments at the same horizontal level are ordered from left to right too. In the written code the various groups are separated by parentheses, the elements of a group by commas."[9, p. 1179].

In the case where a perfect squared rectangle is square, i.e. a perfect squared square, it is necessary to introduce a further rule, that is, in addition to having the largest corner square in the top left corner, the larger of the two boundary squares adjacent to the corner square, go to the right of it. These two elements are the first and the second listed elements in the bouwkampcode. In the case of simple perfect squared squares (SPSSs) the code as just described is chosen as the canonical representative of the eight possible orientations of the squared square[**?**, p (i)].

In the case of CPSSs, which is the concern of this paper, there is the issue of the added complication of the canonical orientation of the smaller squared subrectangle(s) to consider. Each isomer will have a different bouwkampcode, we need to select one as the canonical representative, and as the existing bouwkampcode rules only operate on the first two elements, they will not distinguish CPSS isomers.

A second issue that needs to be resolved with bouwkampcode is the duplication that can result when bouwkampcode is produced for squared rectangles which have a cross. If a squared rectangle is crossed, there are two possible ways of producing the bouwkampcode. If a cross exists in a squared rectangle then there are two horizontal segments which are at the same horizontal level and meet at a point. The bouwkampcode can treat them as either two horizontal segments, or they can be combined into one. The two different bouwkampcodes for the same crossed squared rectangle can result from two different graphs, If different crossed bouwkampcodes describing the same rectangle dissection are not identified and the duplicate bouwkampcode not removed, the squared rectangle enumeration count will be inflated. This issue was highlighted by Gambini.[26, pp.22-24].

Bouwkamp invented bouwkampcode[9] after Brooks, Smith, Tutte and Stone (BSST) wrote their 1940 paper[42]. BSST noted the many-to-one correspondence between p-nets and squared rectangles where there is a zero current, or when two vertices belonging to the same face have equal potential, which in both cases results in a cross in the squared rectangle. They introduced the "normal form" of a p-net which then made the correspondence one-to-one by removing any zero current edges and identifying the nodes of equal potential [42, p.320]. The normal form of a p-net can be encoded unambiguously by using a variation of bouwkampcode discussed in the next section.

3.8 Tablecode and the CPSS canonical representative

A further modification to bouwkampcode can be made. If we form bouwkampcode according to the stated rules, then strip away the parentheses and replace the commas with white space we have a new form of bouwkampcode, due to J.D. Skinner, called tablecode[29]. From tablecode the squared rectangle can always be reconstructed in the same manner as is done with bouwkampcode. Crossed squared rectangles are no longer a source of potential duplication. Removing the parentheses allows only one tablecode to be produced for each dissection, cross or no cross.

With tablecode we also augment the element list by inserting three additional fields into the code at the beginning of the string, that is the order, the width and the height, all separated by spaces. We can also extend the definition of bouwkampcode (or tablecode) by including even more fields. The most useful is an ID field. When more than one CPSS has the same size, it is easier to identify a particular dissection by it's ID rather than having to construct the dissection from the code. IDs are made by concatenating the CPSS size with a letter of the alphabet. We use lowercase alphabet letters for CPSSs and uppercase for SPSSs. For two CPSSs of the same size, the one with the numerically lower tablecode is given the lower alphabet letter. Other extended bouwkampcode (or tablecode) fields are the discoverer's initials, the year of discovery and the number of isomers of that CPSS.

The issue of the canonical orientation of the smaller squared subrectangle in a CPSS can also be solved by using tablecode. The method used by the author is to encode all the isomers of a CPSS by orienting the subrectangle(s) of the CPSS in all possible ways, and orienting each CPSS isomer in all 8 orientations of the square, then producing a tablecode for each of those orientations. Next, for each tablecode, pad each of it's elements with leading zeros so that the number of digits of each element matches the number of digits of the CPSS width field. The zero padded element sizes of each isomer are then concatenated together to form a collection of tablecode isomer strings. The string belonging to the collection which is lexicographically the highest is used to select the corresponding non-zero padded tablecode as the canonical representative of the CPSS and its isomers. The zero padding of element values ensures the lexicographically highest string is also numerically highest. This method is consistent with the earlier bouwkampcode rules[**?**, p (i)] and eliminates any duplicate tilings. Please see Figure 1 on page 2 for examples of a CPSS bouwkampcode and tablecode in canonical form. By selecting the lexicographically and numerically highest tablecode string from the 8 orientations of each CPSS isomer we can also put the isomers into a canonical form.

3.9 Generating graphs with plantri

The graphs used to produce squared squares are generated by a program called *plantri*. *Plantri* is a program that generates certain types of graphs that are embedded in the sphere, so that exactly one member of each isomorphism class is output. Isomorphisms are defined with respect to the embeddings. The program is exceptionally fast and is suitable for the production of large numbers of graphs.[15]

The mathematics and implementation of *plantri* are a collaboration between Gunnar Brinkmann and Brendan D. McKay. McKay distributes the *plantri* generator on his website [15]. Brinkmann has collaborated with O. Delgado Friedrichs, S. Lisken, A. Peeters and N. Van Cleemput to make available a version of *plantri* called *CaGe* (the Chemical and abstract Graph environment), which is a mathematical software package that is intended to be a service to chemists as well as mathematicians, it is designed for 2D and 3D interactive viewing of the graphs it produces [24].

The planar graphs used to produce square tilings are generated in 2 main steps; firstly, it follows from the work of Steinitz [46] that every triangulated sphere can be reduced to the boundary of the tetrahedron by a sequence of edge contractions. In other words, the boundary of the tetrahedron is the only irreducible triangulation of the sphere from which every n-vertex triangulation can be obtained by a suitable sequence of vertex splits. The program *plantri* implements this procedure and allows for a fast enumeration of triangulations of the sphere.[35]

Secondly, general simple plane graphs are produced from the triangulations by the removal of one edge at a time. This is done within specified lower bounds on the minimum degree, the vertex connectivity, the number of edges and if required, an upper bound on the maximum face size[37].

Efficient generation of graphs requires that duplicate graphs (isomorphs) not be produced. The method used for isomorph rejection is the “canonical construction path” method introduced by McKay [37]. Details are in [14]. This method is implemented in *plantri*. Essentially, the program chooses one of the sequences of expansions by which each graph can be made, then rejects any graph made by other sequences. An expansion means replacing some small subgraph by another, usually larger, subgraph under specified conditions. Those graphs not rejected then comprise exactly one member of each isomorphism class.

3.10 2-connected planar graphs and compound dissections

In the theory of squared rectangles developed by Brooks, Smith, Stone and Tutte[42], the dissections of squared rectangles correspond to electrical flows on 2-connected and 3-connected planar graphs embedded in the sphere with one edge distinguished. The 3-connected graphs correspond in most cases to simple dissections, and only have one embedding in the sphere, while 2-connected graphs have multiple embeddings in the sphere, each of which corresponds to a different compound isomer dissection.

A 2-connected planar graph produces compound dissections. More recent proofs of this and other related results are given by Blander and Lo [8]. If a graph has vertices of degree 2 then it will always produce imperfect tilings. By Kirchhoffs current law, the current into the vertex will equal the current coming out. Currents correspond to the sizes of squares in network with unit resistances so the 2 corresponding squares will be of the same size, and hence the dissection is imperfect. It follows that the enumeration of compound perfect squared squares (CPSSs) using electrical network theory will require the production of 2-connected planar embeddings with no vertex of degree 2.

Graphs with no vertices of degree 2 are known as *homeomorphically irreducible* graphs. Unlabelled homeomorphically irreducible 2-connected graphs were counted by T.R.S. Walsh in 1982 using an enumeration tool developed by R.W. Robinson [55]. In 2007 Gagarin, A. and Labelle, G. and Leroux, P. and Walsh, T published [25, p27] and gave counts of unlabelled planar 2-connected graphs. They also gave a formula for 2-connected homeomorphically irreducible planar graphs[25, p32].

Table 4: Homeomorphically irreducible 2-connected embedded planar graphs produced by plantri to enumerate CPSSs to order 29

Node Count V in rows and *Face Count* F in columns, *Edge Count* E = V + F - 2 , in diagonals.														
	6	7	8	9	10	11	12	13	14	15	16	17	18	19
6	1	1												
7	-	3	7	2										
8	-	-	35	60	47	12								
9		-	-	307	647	652	325	59						
10		-	-	-	3 395	7 647	9 582	6 654	2 442	368				
11			-	-	-	38 876	94 278	136 628	121 204	64 232	18 916	2 363		
12			-	-	-	-	468 211	1 192 511	1 937 266	2 049 784	1 409 199	607 746	150 161	16 253
13				-	-	-	-	5 787 837	15 371 597	27 294 367	33 135 263	27 605 162	15 550 020	5 669 267
14				-	-	-	-	-	73 232 219	201 223 550	384 201 336	520 501 148	504 051 385	
15					-	-	-	-	-	944 081 828	2 670 262 417	5 415 258 877		
16					-	-	-	-	-	-	do this!			

Dual graphs where V $>$ F do not need to be produced as they produce the same CPSSs as the graph classes where V $<$ F except for a rotation of the CPSS by 90 degrees. These cells have a dash (-). Each order n of CPSS is enumerated by processing all graph classes in the table with the same edge count E where E - 1 = order n. Table cells where 2F $>$ 3V correspond to non-planar graphs are so are not produced by plantri, similarly for the duals of those graphs where 3F $<$ 2V. Graph classes where 2F = 3V are triangulations. These graphs are 3-connected so they and their duals the cubic graphs, (3F = 2V) are also not produced for this class of 2-connected graphs.

3.11 Counts of CPSSs to order 29

Table 5: Number of Compound Perfect Squared Squares (CPSSs) to Order 29 (2012)

Order	CPSSs	CPSS Isomers
24	1	4
25	2	12
26	16	100
27	46	220
28	143	948
29	412	2308

CPSSs can be counted in two ways. Firstly we count *the number of compound perfect squared squares of order n up to symmetries of the square and its squared subrectangles OEIS A181340* [4], this includes only one representative from both the CPSS class and the CPSS isomer class. This is how CPSSs have been counted to date in the literature.

We introduce a second count, that is *the number of compound perfect squared squares up to symmetries of the square*; *OEIS A217155* [38], this count is the number of members of the CPSS isomer class and includes all the symmetries of any dissected subrectangles, (but not the 8 symmetries of the dissected square).

All the other isomers of a given CPSS isomer can easily be found by examining all the different ways in which sub-rectangle(s) can be oriented within the squared square dissection. The isomers derived geometrically are a useful check on the enumeration of CPSS produced from graphs. The isomer count for a particular CPSS corresponds to all the possible embeddings of the underlying 2-connected graph.

3.12 Bouwkampcode listing of CPSSs, order 24 to order 27

order	sizeID	bouwkampcode	isomers	author	year(s)
24	175a	(81,56,38)(18,20)(55,16,3)(1,5,14)(4)(9)(39)(51,30)(29,31,64)(43,8)(35,2)(33)	(4)	THW	1948
25	235a	(124,111)(43,35,33)(56,38,30)(2,31)(8,29)(81)(18,20)(60)(55,16,3)(1,5,14)(4)(9)(39)	(4)	PJF	1964
25	344a	(147,108,89)(27,62)(100,8)(35)(86,61)(97)(25,136)(111)(56,41)(17,24)(40,14,2)(12,7)(31)(26)	(8)	PJF	1978
26	288a	(136,72,80)(64,8)(88)(67,60,41,32)(120)(16,25)(3,13)(36,27)(4,21)(38,29)(17)(65)(9,56)(47)	(4)	CJB	1971?
26	360a	(207,153)(63,90)(53,36,55,54,9)(45,27)(17,19)(117)(42,26,2)(99)(24,52)(16,34)(58)(6,46)(40)	(4)	E_L	1969?
26	360b	(207,153)(63,90)(68,40,36,54,9)(45,27)(4,7,25)(117)(28,13,3)(10)(15,8)(99)(33)(85,26)(59)	(4)	CJB	1971?
26	384a	(205,179)(80,99)(88,63,54)(9,125)(25,47)(48,28,23)(91,22)(5,18)(20,13)(69)(7,24)(58,17)(41)	(8)	PJF	1962
26	429a	(264,165)(63,102)(24,39)(9,15)(3,6)(95,100,72)(162)(28,44)(70,25)(20,65,27,16)(11,49)(45)(38)	(4)	PJF	1964
26	440a	(250,190)(80,110)(81,45,54,70)(50,30)(36,9)(140)(27,19,17)(120)(2,15)(8,13)(109,38,5)(33)(71)	(4)	CJB	1971?
26	480a	(280,200)(80,120)(116,103,101,40)(160)(2,99)(45,60)(84,32)(52,25)(7,16,37)(3,4)(27,1)(5)(21)	(4)	JDS	1990?
26	483a	(247,236)(100,136)(41,24,31,62,89)(17,7)(12,26)(56,2)(14)(40)(35,27)(8,147,61)(139)(25,111)(86)	(4)	DFL	1979
26	492a	(255,125,55,57)(53,2)(59)(17,25,11)(3,56)(14)(142)(39)(95)(111,96,36,12)(24,225)(60)(15,141)(126)	(4)	THW	1951
26	493a	(218,135,65,75)(55,10)(85)(15,40)(150)(125)(131,87)(67,208)(17,23,47)(11,6)(5,24)(16)(144,3)(141)	(4)	JDS	1990?
26	500a	(195,193,112)(43,29,40)(19,10)(9,1)(41)(38,5)(33)(72,98,135)(125,70)(55,87)(61,37)(180)(172)(148)	(16)	PJF	1964?
26	512a	(202,128,182)(74,54)(87,149)(130,45,34,67)(11,23)(44,12)(35)(92,62)(41,3)(38)(30,181)(180,29)(151)	(8)	DFL	1979
26	608a	(231,194,183)(11,172)(205)(118,113)(5,108)(123)(44,43,85)(1,42)(209,41)(168)(20,27,61)(136,7)(34)(95)	(16)	WTT	1940
26	612a	(289,154,105,64)(28,36)(13,15)(7,29)(49,69)(22)(51)(203)(63,6)(57)(120)(153,136)(68,255)(17,187)(170)	(4)	CJB	1971?
26	638a	(229,232,177)(55,122)(226,3)(223,67)(189)(183,102,92,72)(39,150)(111)(31,23,38)(81,21)(8,15)(60)(53)	(8)	PJF	1964?
26	648a	(378,270)(108,162)(153,128,151,54)(216)(73,55)(32,119)(117,36)(87)(24,12)(16,69)(17,7)(3,13)(10)(40)	(4)	JDS	1990?
27	256a	(118,76,62)(14,48)(56,34)(22,60)(64,54)(40,38)(51,47)(10,84)(74)(8,39)(35,11,5)(1,7)(6)(24)	(4)	JDS	1990?
27	324a	(189,135)(54,81)(76,60,39,41,27)(108)(37,2)(43)(16,44)(59,33)(29,8)(51)(12,31,1)(30)(26,7)(19)	(4)	JDS	1990?
27	325a	(196,129)(67,62)(5,57)(69,37,39,71,52)(35,2)(41)(32,77)(60,9)(58,13)(6,21,8)(15)(49)(45)(36)	(4)	PJF	1962
27	345a	(133,104,108)(100,4)(57,30,25)(62,71)(8,17)(27,3)(11)(2,15)(13)(39,73)(53,9)(44,141,34)(107)(97)	(8)	A& P	2010
27	357a	(197,160)(37,27,44,52)(10,17)(90,75,72,7)(49,19)(11,41)(30)(16,104)(88)(35,40)(70,20)(50,5)(45)	(4)	JDS	1990?
27	360a	(208,152)(56,96)(67,44,49,64,40)(24,112)(28,16)(11,38)(27)(88)(37,25,5)(33)(65)(12,13)(48,1)(47)	(4)	CJB	1970?
27	408a	(264,144)(63,81)(30,33)(15,66)(27,3)(51)(82,80,74,55)(117)(19,36)(6,70,17)(22,64)(62,20)(53)(42)	(4)	PJF	1962
27	440a	(253,187)(77,110)(81,44,51,66,11)(55,33)(37,7)(143)(30,28)(121)(8,20)(106,36,6)(14)(2,18)(16)(70)	(4)	CJB	1970?
27	441a	(249,192)(76,116)(108,90,51)(36,40)(31,16,4)(13,27)(23,133)(15,1)(14)(110)(42,48)(84,24)(60,6)(54)	(4)	JDS	1990?
27	441b	(249,192)(92,100)(108,90,51)(16,76)(39,61)(67)(17,22)(42,48)(12,5)(88)(84,24)(7,81)(74)(60,6)(54)	(4)	JDS	1990?
27	447a	(255,192)(63,55,74)(36,19)(108,90,92,28)(93)(64)(42,48)(100,39,17)(5,88)(84,24)(22)(61)(60,6)(54)	(4)	JDS	1990?
27	460a	(197,127,136)(118,9)(76,69)(115,82)(15,17,37)(68,8)(21,2)(19)(39,1)(38)(83,35)(33,49)(180)(148)(132)	(4)	E_L	1969?
27	468a	(273,195)(78,117)(99,71,66,76,39)(156)(45,21)(31,40)(11,19,46)(24,8)(27)(96,3)(34)(25,84)(73)(59)	(4)	JDS	1990?
27	468b	(273,195)(78,117)(99,81,56,76,39)(156)(25,31)(11,19,46)(21,45,40)(34,8)(27)(96,3)(24)(74)(73)(69)	(4)	JDS	1990?
27	596a	(305,291)(124,86,81)(195,110)(5,76)(91)(181,53)(20,56)(128,36)(96,54,45)(92)(21,24)(42,12)(30,3)(27)	(16)	JDS	1990?
27	599a	(341,258)(38,26,23,43,128)(3,20)(12,17)(45,5)(85)(144,104,138)(213)(70,34)(52,120)(114,30)(84,16)(68)	(4)	DFL	1979
27	600a	(333,267)(66,201)(94,64,82,159)(46,18)(100)(78,16)(62)(69,132)(108,51)(95,35,10)(25,85)(6,63)(60)(57)	(4)	PJF	1962
27	616a	(350,266)(112,154)(87,64,101,98)(70,42)(27,37)(196)(66,21)(17,10)(168)(7,52,89)(45)(113,50)(13,76)(63)	(4)	CJB	1970?
27	618a	(327,291)(105,186)(137,121,69)(99,75)(16,11,28,66)(4,7)(1,3)(154)(10)(24,51)(38)(45,141)(123)(104)(96)	(4)	PJF	1962
27	624a	(335,289)(108,181)(105,60,44,64,62)(16,28)(45,19,12)(2,68,100)(66)(7,33)(26)(184,25)(159)(27,154)(127)	(8)	A& P	2010
27	627a	(352,275)(144,131)(150,135,67)(16,11,17,87)(5,6)(208,3)(24)(23)(47)(15,55,65)(134)(125,40)(85,10)(75)	(4)	JDS	1990?
27	636a	(321,315)(6,141,168)(59,48,85,135)(11,37)(70)(44,78)(80,34)(180,69,27)(42,153)(46,66)(111)(106,20)(86)	(4)	DFL	1979
27	645a	(354,291)(108,183)(163,152,39)(33,6)(27,87)(60)(12,171)(159)(11,71,70)(128,46)(32,14)(1,69)(18,68)(50)	(4)	PJF	1962
27	648a	(333,315)(18,42,108,147)(116,121,90,24)(66)(225,39)(76,35,5)(46,80)(186)(19,16)(3,25,34)(22)(123)(114)	(4)	PJF	1962
27	648b	(405,243)(118,125)(44,67,7)(132)(129,87,120,79,34)(11,56)(45)(164,16)(57,30)(148)(27,3)(123)(114,15)(99)	(4)	A& P	2010
27	652a	(337,315)(22,69,87,137)(135,78,99,47)(61,55)(37,50)(57,21)(36,84)(6,73,13)(67)(200)(180,48)(140)(132)	(4)	JDS	1990?
27	688a	(373,315)(58,102,155)(135,78,99,69,50)(6,43,53)(19,37)(88)(57,21)(70,10)(218)(36,84)(180,48)(158)(132)	(4)	JDS	1990?
27	690a	(375,315)(70,88,157)(135,78,99,53,10)(43,37)(19,69)(6,50)(102)(57,21)(58,218)(36,84)(180,48)(160)(132)	(4)	JDS	1990?
27	690b	(375,315)(73,67,175)(135,78,99,50,13)(6,61)(37,55)(87)(69,47)(57,21)(36,84)(22,200)(180,48)(178)(132)	(4)	JDS	1990?
27	795a	(299,212,284)(87,125)(53,231)(170,216)(178)(124,46)(78,280,176,137)(202)(67,70)(104,44,28)(16,76,3)(73)(60)	(8)	PJF	1970
27	804a	(348,201,255)(147,54)(309)(131,148,216)(114,17)(97,68)(240,123,129,101)(211)(28,73)(117,6)(111,52)(7,66)(59)	(8)	DFL	1979
27	820a	(376,205,239)(171,34)(273)(174,237,136)(101,308)(111,63)(48,124,120,109)(159)(11,98)(44,87)(83,41)(1,43)(42)	(4)	DFL	1979
27	824a	(383,273,168)(100,68)(32,36)(85,43,4)(40)(171,102)(42,1)(41)(22,146)(124)(206,177)(116,55)(325)(29,264)(235)	(8)	DFL	1979
27	825a	(372,251,202)(47,25,28,102)(22,3)(31)(2,55,12)(253)(43)(200)(240,132)(108,156,321)(213,87,48)(39,165)(126)	(4)	PJF	1962
27	847a	(493,354)(133,113,108)(5,103)(20,98)(75,78)(156,114,82,141)(72,3)(282)(32,50)(42,86,18)(68)(213)(198)(154)	(4)	JDS	1990?
27	849a	(472,194,183)(11,172)(205)(44,43,85)(1,42)(209,41)(168)(377,95)(61,34)(27,7)(20,123,241)(108)(5,118)(113)	(4)	THW	1951
27	857a	(488,369)(119,250)(172,147,73,72,143)(1,71)(74)(12,238)(226)(25,114,82)(197)(30,52)(8,22)(108,6)(14)(88)	(4)	A& P	2010
27	861a	(311,369,181)(100,81)(13,20,48)(6,7)(105,1)(28)(76)(253,58)(195,215,17)(198)(297,151)(136,277)(146,5)(141)	(8)	DFL	1979
27	867a	(490,377)(113,108,61,95)(27,34)(20,7)(136)(5,123)(209,205,194)(259)(11,183)(44,172)(168,41)(1,43)(42)(85)	(4)	THW	1951
27	869a	(428,264,177)(61,116)(26,35)(281,9)(44)(24,92)(68)(160)(188,123,117)(64,170,324)(65,58)(122)(253)(16,154)(138)	(4)	PJF	1970?
27	872a	(495,194,183)(11,172)(205)(44,43,85)(1,42)(209,41)(168)(377,118)(5,108,264)(123)(20,27,61)(136,7)(34)(95)	(4)	THW	1951
27	882a	(532,350)(189,161)(28,133)(211,202,119)(112,105)(238)(231)(54,148)(139,35,37)(33,2)(31,8)(23,39)(71,16)(55)	(4)	CJB	1970?
27	890a	(513,377)(136,123,118)(5,113)(20,108)(209,205,194,34,7)(27)(61)(282)(11,183)(44,172)(168,41)(1,43)(42)(85)	(4)	THW	1951
27	892a	(449,443)(177,266)(223,226)(55,122)(220,3)(217,67)(72,102,92)(150,39)(111)(31,23,38)(81,21)(8,15)(60)(53)	(4)	JDS	1990?
27	904a	(455,449)(223,226)(102,92,111,150)(31,23,38)(81,21)(72,39)(8,15)(60)(53)(122,67)(266)(55,232,3)(229)(177)	(4)	PJF	1970?
27	931a	(342,281,165,143)(67,76)(120,45)(75,28,9)(19,66)(47)(61,216,4)(312)(248,155)(93,178,100)(341)(78,334)(256)	(4)	DFL	1979

3.13 Bouwkampcode listing of CPSSs, order 28, part 1

order	sizeID	bouwkampcode	isomers	author	year	year
28	312a	(128,100,84)(34,50)(28,54,18)(36,16)(86,70)(66)(44,46)(114)(112)(47,39)(8,7,24)(1,6)(51,5)(11)	(8)	GHM		
28	335a	(131,116,88)(28,60)(79,32,18,15)(67,64)(7,8)(14,4)(10,1)(9)(47,78)(19,140,31)(51,16)(35)(109)	(8)	JDS	1993	2003
28	374a	(169,111,56,38)(18,20)(55,16,3)(1,5,14)(4)(9)(39)(88,117)(86,53,30)(23,66,29)(33,43)(146)(119)	(8)	PJF	1962	
28	427a	(233,194)(37,54,103)(20,17)(94,65,56,18)(41,30)(38)(11,19)(138,8)(130)(29,36)(100,16,7)(9,34)(25)	(4)	JDS	1993	2003
28	430a	(234,196)(61,49,86)(57,41,65,71)(12,37)(48,25)(15,26)(148)(4,11)(54,3)(7)(59,6)(44)(125)(85,13)	(4)	A&P	2010	
28	435a	(176,147,112)(65,25,22)(3,19)(17,11)(29,118)(6,5)(24)(23)(116,89)(50,62)(38,12)(74)(27,170,48)(143)	(16)	A&P	2010	
28	444a	(254,190)(55,135)(16,39)(100,96,51,7)(23)(21,41)(72)(57,119)(14,38,44)(90,10)(24)(67,5)(62)(56,6)	(4)	JDS	1993	2003
28	450a	(238,212)(62,53,97)(108,57,73)(9,44)(71)(30,27)(141)(39,34)(3,11,13)(25,8)(17,2)(15)(105)(104,4)	(8)	A&P	2010	
28	457a	(213,144,100)(36,64)(8,28)(132,20)(112)(86,56,71)(30,14,12)(63,181)(2,7,3)(16)(4,15,55)(11)(158)	(4)	A&P	2010	
28	468a	(221,133,54,60)(39,15)(9,7,12,32)(2,5)(23,3)(20)(1,74)(40)(140,33)(107)(117,104)(52,195)(13,143)	(4)	JDS	1993	2003
28	471a	(227,144,100)(36,64)(8,28)(132,20)(112)(114,57,56)(49,195)(2,12,42)(41,11,4,1)(3)(7)(30)(16,146)	(16)	A&P	2010	
28	472a	(207,153,65,25,22)(3,19)(17,11)(6,5)(24)(23)(50,62)(38,12)(74)(143,48)(122)(118,89)(29,176,27)(149)	(8)	A&P	2010	
28	475a	(219,144,112)(32,80)(120,56)(8,72)(64)(126,48,45)(13,77,211)(38,10)(23)(7,16)(36,2)(9)(25)(4,134)	(4)	A&P	2010	
28	488a	(255,233)(111,122)(86,80,89)(26,54)(66,20)(45,144,11)(133)(46)(99)(43,29,40)(19,10)(9,1)(41)(38,5)	(4)	JDS	1993	2003
28	520a	(299,221)(78,143)(97,70,113,65,32)(15,8,9)(7,1)(10)(18,4)(14)(84,156)(27,43)(124)(108,48)(12,72)	(16)	THW	1964	1971
28	532a	(222,183,127)(66,61)(99,84)(5,56)(71)(136,86)(19,37)(1,18)(72)(15,69)(55)(26,88)(50,62)(196)(174,12)	(8)	JDS	1990	1993
28	550a	(271,149,130)(19,15,26,70)(4,11)(165,7)(44)(114)(139,89,43)(86,236)(49,40)(57,69)(1,48)(140)(93,12)	(4)	JDS	1993	2003
28	557a	(317,240)(77,50,113)(19,31)(8,11)(123,129,101,36,13)(10,1)(32)(23)(204)(28,73)(117,6)(111,52)(7,66)	(4)	A&P	2010	
28	565a	(247,152,166)(95,32,11,14)(8,3)(5,18,160)(13)(63)(178,148,79)(239)(30,46,72)(140,52,16)(36,26)(98)	(4)	JDS	1993	2003
28	568a	(262,150,156)(144,6)(99,63)(36,27)(11,16)(95,28,10,2)(8,5)(142,120)(21)(18)(67)(88,56)(218)(22,186)	(4)	PJF	1964	1971
28	569a	(239,151,179)(123,28)(207)(86,96,57)(22,101)(79)(76,10)(66,40)(26,194)(108,99)(168)(15,84)(75,27,6)(21)	(8)	JDS	1993	2003
28	571a	(334,135,102)(33,69)(132,36)(105)(123,9)(114)(121,126,87)(56,181)(59,28)(3,53)(31)(116,5)(111,20)(91,19)	(4)	A&P	2010	
28	576a	(270,195,111)(60,51)(19,32)(24,26,10)(16,13)(45)(175,44)(42)(131)(154,116)(100,206)(38,78)(152,40)(112,6)	(4)	CJB	pre	1990
28	577a	(337,240)(65,62,113)(11,51)(32,25,8)(19)(23,2)(21)(123,129,101,16)(224)(28,73)(117,6)(111,52)(7,66)	(4)	THW	1948	
28	581a	(338,243)(95,148)(129,87,120,97)(44,104)(57,30)(81,60)(27,3)(123)(114,15)(99)(37,41,86)(65,16)(49,4)	(4)	A&P	2010	
28	590a	(258,182,150)(32,118)(120,94)(75,51,69,63)(8,110)(102)(19,101)(33,18)(82)(15,72)(66,9)(57)(212)(191,4)	(4)	A&P	2010	
28	591a	(328,263)(148,115)(127,118,83)(33,82)(35,180,49)(9,69,37,38)(136)(131)(17,19,1)(39)(15,2)(21)(76,8)	(4)	PJF	pre	1982
28	592a	(229,148,215)(81,67)(35,91,156)(130,159,21)(56)(82,65)(66,36,28)(221)(8,20)(37,204)(30,14)(2,18)(16)	(16)	JDS	1990	1993
28	596a	(276,180,140)(40,100)(150,70)(10,90)(80)(167,109)(54,266)(1,53)(58,52)(6,24,75)(153,60,18)(42)(93,9)	(4)	JDS	1993	2003
28	596b	(276,180,140)(40,100)(150,70)(10,90)(80)(171,105)(54,266)(42,60,3)(57)(24,18)(6,58,71)(149,52)(97,13)	(4)	JDS	1993	2003
28	600a	(344,256)(75,76,105)(13,33,29)(28,48)(144,120,73,20)(57)(19,86)(53)(67)(183)(153)(56,64)(112,32)(80,8)	(4)	JDS	1993	2003
28	612a	(312,146,75,79)(71,4)(83)(111,90,16)(99)(21,44,25)(109,23)(19,105)(86)(126,78,108)(54,246)(48,30)(192)	(4)	CJB	pre	1990
28	612b	(357,255)(102,153)(129,91,96,92,51)(204)(41,50)(33,59)(60,36)(7,26)(126,3)(44)(24,19)(35,15)(104)(99)	(4)	JDS	1993	2003
28	612c	(357,255)(102,153)(143,135,130,51)(204)(5,125)(42,40,58)(112,31)(28,3)(2,24,14)(25,22)(10,4)(62)(56)	(4)	JDS	1993	2003
28	630a	(300,160,77,93)(40,37)(21,72)(3,55)(43)(194,9)(19,53)(49,15)(34)(136)(144,102,54)(108,276)(42,60)(186)	(4)	CJB	pre	1990
28	632a	(321,311)(153,158)(178,143)(35,47,92,64,53,5)(163)(133,68,12)(59)(11,42)(44,31)(76,16)(73)(65,3)(62)	(4)	CJB	pre	1990
28	645a	(290,170,185)(155,15)(200)(160,130)(95,60)(260)(225)(50,43,67)(7,36)(57)(11,56)(1,10)(28,9)(19)(88,16)	(8)	CJB	pre	1990
28	656a	(379,277)(102,175)(174,164,143)(79,96)(134,9)(49,22,17)(51,113)(5,108)(103,43,28)(27)(76)(15,13)(2,62)	(4)	A&P	2010	
28	660a	(286,179,82,113)(51,31)(40,104)(19,12,20)(7,5)(2,3)(27,1)(64)(206)(168)(165,121)(99,275)(44,77)(209)	(4)	JDS	1990	1993
28	669a	(303,216,150)(54,96)(12,42)(198,30)(168)(112,73,118)(28,45)(11,17)(111,255)(123)(109,71)(38,33)(144)(8,139)	(4)	A&P	2010	
28	676a	(379,297)(118,179)(129,90,117,43)(16,41,61)(34,9)(25)(80,20)(39,51)(60,200)(24,93)(168)(75)(140)(6,87)	(4)	A&P	2010	
28	684a	(390,294)(148,146)(141,117,132)(54,92)(80,68)(16,38)(24,21,72)(62,22)(162,50)(3,18)(153,15)(152)(33)(112)	(4)	JDS	1993	2003
28	702a	(378,324)(48,53,69,154)(6,42)(37,16)(149,120,115)(85)(79)(15,224)(209)(29,32,59)(175,3)(35)(8,51)(43)	(4)	JDS	1993	2003
28	704a	(395,309)(86,95,128)(180,174,127)(63,32)(31,1)(30,99)(55,69)(182)(168)(18,21,135)(129,39,12)(27,3)(24)	(4)	A&P	2010	
28	712a	(291,240,181)(87,94)(212,28)(108,7)(101)(178,113)(209)(65,48)(260)(243)(72,71,66)(5,61)(1,19,56)(55,18)	(8)	A&P	2010	
28	714a	(423,291)(96,195)(36,60)(163,152,120,24)(45,39)(6,228)(51)(171)(11,71,70)(128,46)(32,14)(1,69)(18,68)	(4)	PJF	1962	
28	732a	(276,207,249)(165,42)(291)(183,93)(90,3)(168)(273)(192,106,76,85)(30,46)(37,48)(86,34,16)(18,70,11)(59)	(8)	PJF	1962	
28	732a	(436,296)(113,183)(43,70)(101,76,120,114,25)(13,30)(21,4)(17)(253)(68)(32,44)(74,27)(20,12)(182)(176)(47)	(4)	PJF	pre	1982
28	741a	(348,259,134)(53,81)(25,28)(18,7)(4,105)(11)(29)(227,61)(166)(210,138)(99,69,108,255)(30,39)(183,27)(156)	(4)	PJF	1962	
28	742a	(422,320)(102,218)(209,101,84,130)(38,46)(80,21)(14,204)(59)(190)(139)(111,54,44)(7,11,26)(3,4)(57)(15)	(16)	PJF	pre	1982
28	753a	(287,249,217)(100,117)(181,68)(144,143)(151,17)(134)(1,323)(145)(285)(89,56)(26,30)(7,15,4)(34)(88,8)(23)	(8)	A&P	2010	
28	756a	(357,262,137)(81,56)(25,31)(57,43,6)(37)(80)(179,59,24)(11,46)(35)(120,20)(100)(189,168)(84,315)(21,231)	(4)	JDS	1990	1993
28	765a	(297,231,237)(225,6)(243)(138,159)(117,21)(96,309)(81,62,100)(19,43)(213)(76,24)(5,25,70)(52,20)(45)(128)	(8)	PJF	1962	
28	765b	(381,202,96,86)(10,76)(106)(40,36)(4,32)(236,88,28)(60)(148)(159,81,84,57)(114,327)(78,3)(87)(225,12)	(4)	PJF	1962	
28	770a	(408,362)(190,172)(199,85,57,67)(47,10)(77)(59,26)(7,40)(33)(70,102)(91,218,90)(38,32)(163,36)(134)(128)	(4)	A&P	2010	
28	779a	(427,352)(75,63,214)(20,43)(198,156,140,8)(28)(5,38)(33)(71)(1,213)(212)(42,114)(154,86)(32,82)(68,18)	(4)	A&P	2010	
28	780a	(455,325)(130,195)(197,161,162,65)(260)(76,84,1)(163)(128,69)(39,25,12)(4,80)(16)(22,3)(19)(59,10)(49)	(4)	JDS	1993	2003
28	782a	(297,281,204)(77,127)(81,227,50)(232,65)(177)(146)(21,274,124,68,63)(253)(17,46)(56,12)(29)(15,60)(150,45)	(8)	JDS	1990	1993
28	783a	(441,342)(99,243)(236,193,111)(75,36)(3,42,198)(39)(156)(44,149)(106,51,47,31,1)(45)(16,15)(60)(4,59)	(4)	CJB	pre	1990
28	792a	(351,297,144)(53,36,55)(17,19)(42,26,2)(24,52)(16,34)(58)(6,46)(40)(207,234)(198,153)(45,288,27)(261)	(16)	CJB	pre	1990
28	792b	(351,297,144)(68,40,36)(4,7,25)(28,13,3)(10)(15,8)(33)(85,26)(59)(207,234)(198,153)(45,288,27)(261)	(16)	CJB	pre	1990
28	792c	(450,342)(144,198)(82,115,127,126)(90,54)(49,33)(252)(16,81,39,12)(216)(27,112)(65)(66)(146)(47,19)(28,103)	(4)	JDS	1990	1993
28	792d	(450,342)(144,198)(149,81,94,126)(90,54)(68,13)(252)(32,27,48)(6,21)(23,8,1)(216)(7)(15)(69)(193,62)	(4)	JDS	1990	1993
28	792e	(450,342)(144,198)(151,81,92,126)(90,54)(70,11)(252)(103)(216)(102,75,44)(147)(27,48)(89,40)(19,29)(49,10)	(16)	A&P	2010	
28	802a	(439,363)(123,240)(143,66,36,28,119,47)(8,20)(30,14)(72,98)(2,18)(16)(77,53)(244)(220)(79,19)(60,199)	(8)	A&P	2010	
28	804a	(357,213,234)(192,21)(255)(201,156)(108,84)(24,315)(45,139,104)(246)(35,26,43)(9,17)(152,23,8)(15,53)(38)	(4)	PJF	pre	1982
28	804b	(492,312)(105,207)(48,57)(27,21)(12,45)(33)(175,152,86,106)(285)(66,20)(43,83)(3,40)(61,160)(137,38)(123)	(4)	CJB	pre	1990
28	805a	(462,202,141)(61,80)(126,118,19)(99)(8,209)(134)(195,144,190,67)(276)(98,46)(83,153)(148,47)(101,44)(13,70)	(4)	JDS	1993	2003
28	807a	(455,352)(103,249)(198,165,195)(43,206)(77,88)(6,37)(157,21,23)(154,44)(19,2)(17,8)(45)(36)(110,11)(99)	(4)	A&P	2010	
28	811a	(435,376)(134,242)(184,176,75)(101,108)(8,200,69)(62,95,193)(192)(131)(50,45)(5,12,28)(48,7)(19)(3,25)	(16)	A&P	2010	
28	811b	(460,351)(109,242)(151,101,152,88,77)(13,28,36)(69,17,2)(15)(50,51)(52,8)(44)(24,218)(200,1)(199,5)	(4)	A&P	2010	
28	812a	(435,377)(174,203)(175,73,71,116)(2,69)(75)(261,29)(8,20,41)(232)(27,44,12)(32)(202)(11,30)(68,19)(49)	(4)	JDS	1993	2003
28	815a	(479,336)(195,75,66)(9,57)(51,33)(18,15)(72)(69)(191,184,104)(52,284)(59,97)(21,38)(53,152)(145,46)(135)	(16)	A&P	2010	
28	816a	(331,253,232)(21,211)(150,124)(204,127)(56,68)(105,45)(15,29,12)(65,146)(17,63)(60)(77,50)(46)(308,81)(281)	(4)	DFL	1979	
28	820a	(492,328)(123,205)(41,82)(186,162,185)(287)(24,86,27,13,12)(1,11)(14)(4,49,143)(142,68)(45)(94)(6,80)	(4)	JDS	1990	1993
28	820b	(492,328)(123,205)(41,82)(190,66,49,89,139)(287)(17,32)(68,15)(47)(39,50)(6,80)(74)(189)(138,52)(34,120)	(4)	JDS	1990	1993
28	824a	(436,388)(140,248)(187,157,92)(132,100)(13,9,14,33,88)(4,5)(17)(19)(201,3)(55)(32,68)(60,188)(164)(143)	(4)	PJF	1962	
28	828a	(381,213,234)(192,21)(255)(201,156,24)(132,84)(339)(45,139,104)(246)(35,26,43)(9,17)(152,23,8)(15,53)(38)	(4)	PJF	pre	1982
28	834a	(455,379)(139,240)(159,98,135,63)(72,66,36,28)(8,20)(61,37)(30,14)(2,18)(16)(95,35)(244)(220)(60,215)	(8)	A&P	2010	
28	840a	(450,390)(180,210)(119,88,123,120)(53,35)(97,22)(270,30)(47,111)(75)(240)(26,21)(5,16)(20,11)(27)(174,18)	(4)	JDS	1993	2003
28	847a	(362,281,204)(77,127)(81,227,50)(177)(232,211)(65,339)(21,124,68,63)(253)(17,46)(56,12)(29)(15,60)(150,45)	(4)	JDS	1993	2003
28	847b	(473,374)(99,275)(168,92,103,209)(36,45,11)(114)(19,17)(8,37)(2,23)(21)(206,6)(43)(77,198)(165,44)(157)	(4)	JDS	1990	1993
28	854a	(466,388)(74,119,195)(29,45)(188,130,123,25)(38,16)(22,82,76)(60)(271)(265)(58,72)(200,32,14)(18,68)(50)	(4)	JDS	1993	2003
28	855a	(460,220,175)(45,130)(180,85)(215)(60,120)(232,155,133)(335)(22,111)(88,89)(163,58,11)(47,51,1)(50,151)(105)	(4)	CJB	pre	1990
28	868a	(465,403)(186,217)(107,95,139,124)(12,83)(119)(279,31)(39,100)(248)(48,38,36)(2,9,25)(10,23,7)(16)(177)	(4)	JDS	1993	2003
28	868b	(465,403)(186,217)(83,51,54,153,124)(32,19)(16,38)(13,22)(128)(60)(279,31)(59,1)(53,101)(248)(5,48)(192)	(4)	JDS	1993	2003
28	872a	(437,267,168)(78,90)(21,45,12)(102)(264,24)(69)(171)(131,118,188)(94,341)(13,105)(144)(35,247)(52,88)(160,36)	(4)	A&P	2010	
28	877a	(471,406)(25,22,131,228)(3,19)(17,11)(6,5)(24)(23)(173,113,166,66)(100,97)(60,53)(82,243)(240,79)(233)	(4)	A&P	2010	

3.14 Bouwkampcode listing of CPSSs, order 28, part 2

order	sizeID	bouwkampcode	isomers	author	year	year
28	1015a	(382,280,188,165)(23,142)(92,119)(372)(261)(199,183)(16,167)(84,177)(215)(363,93)(47,120)(270)(17,30)(219,13)(43)	(16)	WTT	1940	
28	1015b	(593,422)(247,175)(222,164,207)(72,103)(154,116,49)(18,85)(67)(80,41,43)(38,230)(39,2)(37,215)(200,22)(192)	(48)	AHS	1940	
28	1032a	(477,308,247)(138,109)(135,173)(29,80)(116,51)(49,82)(97,38)(258,219)(59,152)(16,33)(132)(115)(156)(399)(39,336)	(8)	A&P	2010	
28	1049a	(560,489)(232,257)(222,177,161)(16,49,328)(45,115,33)(154,103)(82)(267)(51,52)(197)(149,41,14,1)(13,40)(27)	(8)	A&P	2010	
28	1056a	(522,292,242)(73,169)(269,23)(96)(265)(199,130,193)(154,115)(36,25,69)(380)(9,16)(2,7)(33,5)(28)(6,341)	(32)	A&P	2010	
28	1057a	(396,307,354)(260,47)(401)(276,120)(69,51)(18,105,188)(87)(385,83)(302,190,180)(10,48,122)(112,50,38)(12,74)	(8)	A&P	2010	
28	1064a	(431,379,254)(132,122)(10,112)(52,172,155)(142)(285,198)(38,74)(2,36)(144)(17,138)(189)(110)(87,111)(392)(348,24)	(8)	JDS	1993	2003
28	1069a	(545,524)(188,336)(213,165,167)(163,2)(357)(131,82)(49,33)(209,72,55)(196)(180)(18,37)(71,1)(19)(56)(66,5)	(16)	A&P	2010	
28	1071a	(588,483)(225,258)(230,157,201)(81,144)(111,147)(73,84)(282)(253,50)(20,21,43)(19,1)(22)(219,36)(69)(65)(183)	(4)	PJF	1962	
28	1073a	(465,349,259)(90,169)(360,79)(248)(111,133,221)(89,22)(67,88)(156)(33,62,153)(135,174)(364,29)(91)(252,39)(244)	(16)	WTT	1948	
28	1075a	(427,359,289)(70,219)(280,149)(215,212)(368)(56,436)(162,53)(109)(271)(199,105,64)(33,31)(2,29)(8,27)(94,19)	(16)	A&P	2010	
28	1076a	(492,332,252)(83,169)(199,130,3)(86)(255)(69,36,25)(9,16)(269,223)(2,7)(33,5)(28)(184,145)(400)(46,361)	(32)	A&P	2010	
28	1078a	(593,485)(170,315)(254,277,62)(232)(160,155)(231,23)(208,75,17)(58,191)(133)(5,8,19,123)(118,44,3)(11)(30)	(8)	A&P	2010	
28	1080a	(510,270,300)(240,30)(330)(341,289,120)(450)(137,152)(97,64,95,85)(33,31)(10,144,53,15)(2,134)(132)(38,129)	(4)	JDS	1993	2003
28	1089a	(585,504)(213,126,165)(251,202,132)(87,39)(48,156)(372,108)(47,25,28,102)(22,3)(31)(264)(2,55,12)(253)(43)	(4)	PJF	1962	
28	1089b	(660,429)(231,198)(33,165)(248,268,137,139,132)(297)(135,2)(141)(181,67)(47,161,60)(56,71,8)(149)(114)(101,15)	(4)	JDS	1993	2003
28	1092a	(585,507)(234,273)(138,75,43,54,119,156)(32,11)(65)(63,44)(228)(201)(351,39)(312)(168,25,8)(17,59,160)(42)	(4)	JDS	1993	2003
28	1093a	(593,500)(147,353)(284,255,54)(201)(97,114,245)(123,129,101)(216,68)(148,17)(28,73)(131)(117,6)(111,52)(7,66)	(8)	A&P	2010	
28	1108a	(593,515)(78,124,313)(231,178,216,46)(170)(53,125)(87,299)(284)(176,137)(212)(67,70)(104,44,28)(16,76,3)(73)	(8)	PJF	pre	1982
28	1113a	(639,474)(165,309)(275,238,204,87)(51,36)(15,21)(66)(330)(270)(113,125)(199,76)(123,43,23)(14,111)(20,3)(17)	(4)	PJF	1962	
28	1115a	(582,533)(48,95,135,132,123)(1,47)(279,304)(142)(9,114)(36,105)(102,33)(69)(187,345)(254,25)(229,100)(29,158)	(4)	A&P	2010	
28	1116a	(527,305,161,123)(38,85)(144,55)(8,17,60)(52,11)(2,15)(13)(28)(140)(346,103)(243)(279,248)(124,465)(31,341)	(4)	JDS	1990	1993
28	1116b	(651,465)(186,279)(242,261,241,93)(372)(96,145)(223,19)(204,76)(128,44)(39,5)(28,43,79)(6,22)(45)(7,36)	(4)	JDS	1993	2003
28	1131a	(651,480)(171,309)(261,168,164,124,105)(72,33)(71,53)(342)(4,41,119)(135,37)(18,107)(89)(78)(219,42)(197)(196)	(4)	A&P	2010	
28	1132a	(658,474)(184,131,159)(53,38,40)(12,147)(15,23)(52)(270,264,246,122,8)(31)(83)(352)(18,228)(72,210)(204,66)	(4)	JDS	1993	2003
28	1134a	(684,450)(243,207)(36,171)(251,138,142,153)(144,135)(70,45,23)(19,123)(306)(297)(42)(25,20)(62)(95)(199,52)(185)	(4)	JDS	1990	1993
28	1137a	(593,544)(49,145,122,228)(332,310)(23,99)(168)(92,7)(85,150)(280,65)(22,54,234)(215)(212,110,32)(86)(102,8)	(16)	PJF	1962	
28	1138a	(478,302,358)(246,56)(414)(172,179,127)(57,189)(52,132)(165,7)(158,80)(401)(216,198)(323)(30,168)(150,54,12)(42)	(8)	JDS	1993	2003
28	1140a	(625,515)(202,176,137)(231,178,216)(67,70)(104,44,28)(124,78)(16,76,3)(73)(60)(46,345)(53,125)(87,299)(284)	(4)	JDS	1990	1993
28	1145a	(657,488)(274,214)(288,264,105)(60,154)(182,163,94)(69,62,117)(7,55)(40,224)(19,220)(201)(200,72,16)(56)(172)	(4)	A&P	2010	
28	1151a	(644,507)(184,164,159)(315,282,47)(5,154)(20,149)(122,129)(115,7)(108,331)(57,225)(223)(192,99,24)(81)(93,6)	(4)	A&P	2010	
28	1152a	(672,480)(192,288)(218,228,84,111,127,96)(57,27)(384)(30,92,16)(143)(87)(25,67)(120,98)(88,252)(210)(22,164)	(4)	A&P	2010	
28	1155a	(645,510)(135,375)(228,113,154,285)(61,52)(11,143)(9,25,29)(54,16)(37,4)(33)(70)(282)(105,270)(225,60)(213)	(4)	CJB	pre	1990
28	1155a	(700,455)(245,210)(35,175)(263,164,176,237,140)(315)(120,44)(32,83,61)(76)(80,218)(25,58)(192,71)(50,171)(138)	(4)	JDS	1990	1993
28	1157a	(593,564)(97,114,353)(309,216,68)(148,17)(131)(147,348)(255,54)(123,129,101)(201)(28,73)(117,6)(111,52)(7,66)	(8)	JDS	1993	2003
28	1164a	(593,571)(22,54,145,122,228)(261,212,110,32)(86)(23,99)(102,8)(94)(168)(92,7)(85,150)(49,359)(310)(280,65)	(4)	JDS	1993	2003
28	1164b	(684,480)(171,309)(33,138)(261,168,164,124)(72,375)(71,53)(4,41,119)(135,37)(18,107)(89)(78)(219,42)(197)(196)	(4)	A&P	2010	
28	1170a	(704,466)(176,290)(62,114)(247,223,180,74,42)(32,10)(414)(106)(286)(52,68,103)(219,28)(80)(33,35)(31,2)(140)	(4)	A&P	2010	
28	1170b	(704,466)(176,290)(62,114)(267,203,180,74,42)(32,10)(414)(106)(286)(80,123)(199,68)(52,28)(13,15)(11,2)(140)	(4)	A&P	2010	
28	1175a	(507,316,352)(280,36)(388)(266,41,38,73,89)(3,35)(44)(12,23,57,16)(45,11)(241,144)(34)(136)(97,435)(402)	(4)	JDS	1993	2003
28	1186a	(671,515)(202,176,137)(67,70)(231,178,216,46)(104,44,28)(170,78)(16,76,3)(73)(60)(391)(53,125)(87,299)(284)	(4)	PJF	pre	1982
28	1200a	(700,500)(200,300)(277,152,144,227,100)(400)(61,83)(97,55)(2,59)(57)(37,273)(82,15)(67,101)(223,54)(169,34)	(4)	JDS	1993	2003
28	1208a	(615,593)(22,54,234,283)(280,215,110,32)(86)(102,8)(94)(381,49)(65,150)(332)(168,92,85)(7,228)(99)(145,23)	(4)	A&P	2010	
28	1211a	(431,379,401)(52,305,22)(423)(230,253)(108,99,23)(160,281,140)(15,84)(75,27,6)(21)(48)(176,387)(367)(246,35)	(4)	A&P	2010	
28	1224a	(575,313,106,84,146)(22,62)(128)(88,120)(184,32)(152)(208,105)(136,305)(175,33)(169)(306,269)(148,27)(501)(37,380)	(4)	A&P	2010	
28	1224b	(714,510)(204,306)(258,207,167,184,102)(408)(70,97)(66,118)(57,120,30)(100)(14,52)(252,6)(63)(73,38)(208)(183)	(4)	JDS	1993	2003
28	1224c	(714,510)(204,306)(286,270,260,102)(408)(10,250)(85,79,116)(224,62)(55,7)(50,29)(48,44)(21,8)(124)(4,111)	(4)	JDS	1993	2003
28	1225a	(632,593)(39,237,317)(167,172,134,198)(38,96)(162,5)(157,58)(187,166,82)(154)(2,315)(84)(264,55)(21,229)(209)	(4)	PJF	pre	1982
28	1229a	(621,608)(169,174,265)(259,206,156)(237,88)(83,91)(53,153)(171)(356)(133,79,100)(54,25)(4,249)(29)(216)(215,22)	(4)	A&P	2010	
28	1231a	(623,608)(101,140,103,264)(159,169,209,86)(187)(37,66)(148,29)(95)(149,10)(139,40)(87,162)(359)(335)(12,75)(300)	(4)	JDS	1993	2003
28	1236a	(721,515)(206,309)(306,228,290,103)(412)(101,127)(65,225)(209,67,30)(7,44,50)(37)(32,160)(142,6)(48,8)(40)	(4)	JDS	1993	2003
28	1240a	(632,608)(148,187,273)(159,169,209,95)(66,29)(37,140)(149,10)(103)(101,86)(139,40)(87,162)(359)(344)(12,75)(300)	(4)	JDS	1993	2003
28	1272a	(742,530)(212,318)(278,304,266,106)(424)(43,93,130)(252,26)(226,99,5)(48)(46,2)(62,33)(29,4)(134)(127,18)	(4)	JDS	1993	2003
28	1272b	(742,530)(212,318)(278,304,266,106)(424)(59,77,130)(252,26)(226,83,21)(62,18)(46,49)(143,2)(48)(45,4)(134)	(4)	JDS	1993	2003
28	1284a	(749,535)(214,321)(296,160,156,244,107)(428)(4,64,88)(104,60)(59,65)(41,291)(89,15)(74)(106)(239,57)(182,38)	(4)	JDS	1993	2003

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