GRAPH THEORY AND RELATED TOPICS

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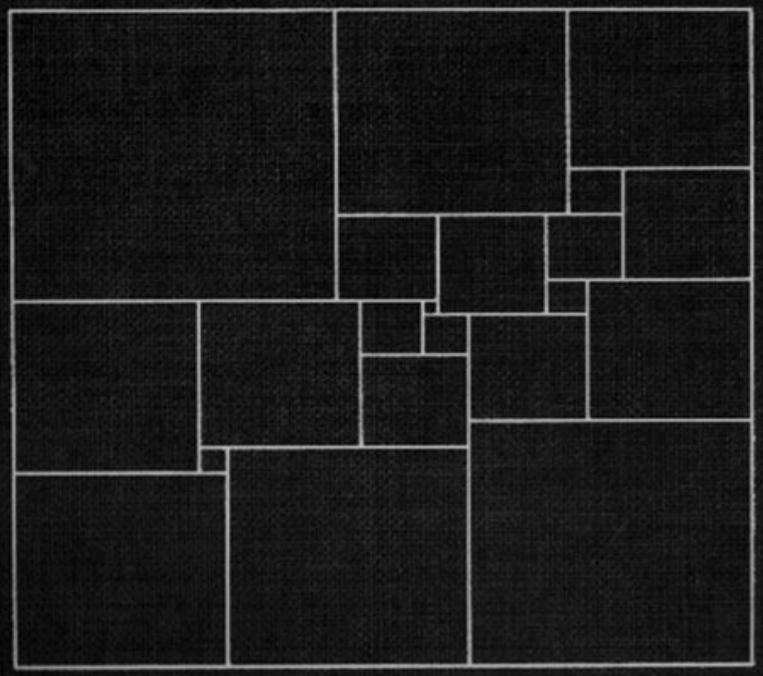

Squaring Rectangles and Squares

A Historical Review with Annotated Bibliography

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The subject of this historical note is the dissection of rectangles and squares into unequal squares, with ancillary material. The treatment is first a brief general historical review, followed and supplemented by an annotated bibliography. Some terminology customarily used in this field is initially set down.

A rectangle (square) dissected into squares is called a *squared rectangle* (*square*). The component squares are the *elements* and the number of elements is the *order*. The dissection is *perfect* if no two elements are equal, otherwise it is *imperfect*. The dissection is *simple* if it does not contain any subset of elements (more than one and less than all) arranged in a rectangle (square), otherwise it is *compound*. Note that reference to a simple or compound squared rectangle does not imply that it is necessarily perfect. Figures 3 and 5–9 illustrate some compound perfect squares and Fig. 10 a simple perfect square; these will be referred to later.

The first explicit mention of division into unequal squares that has been found was in 1925. Prior to this time Dehn had published, in 1903, a study of the division of a rectangle into rectangles [1]. The treatment is analytical with the only noteworthy result, for which the paper has been frequently cited,

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that a rectangle cannot be divided into squares unless its sides are commensurate. There is no mention of division into unequal squares. Perhaps the puzzlists Sam Loyd and H. E. Dudeney thought of dividing a square into unequal squares. Both, between 1903 and 1925, presented the patchwork quilt puzzle, the solution of which required the division of a square into squares, not all unequal [2, 3, 53]. Dudeney also presented a puzzle of dividing a given square into unequal squares and one (given) rectangle [4].

The first examples of rectangles divided into unequal squares were given by Moroń in 1925 [5]. He first states that S. Ruziewicz had proposed the problem, "Can one assemble a rectangle from different squares?" (No publication of this question has been located.) The two examples "of rectangles which may be made up of different squares" are given, without indicating how they were obtained, to answer this question in the affirmative. Moroń's figures are reproduced here in Fig. 1, numbered I and II for later reference. Rectangle I is 33 × 32 in size and is divided into nine unequal squares; rectangle II is 65 × 47 and has ten squares. Moroń's figures introduce two conventions which have been followed; expressing the sides of the component squares in integers without any common divisor (unless some reason requires otherwise), and writing the length of the side of a square inside the square.

Other perfect squared rectangles can be formed from a given one by adding a square of the same length of side to either side, and continuing this process alternately. Moroń gives a general formula for the ratio of sides of the rectangles in this infinite series of trivially compounded squared rectangles [this ratio, incidentally, approaches τ (sometimes also called ϕ), the golden ratio, as a limit].

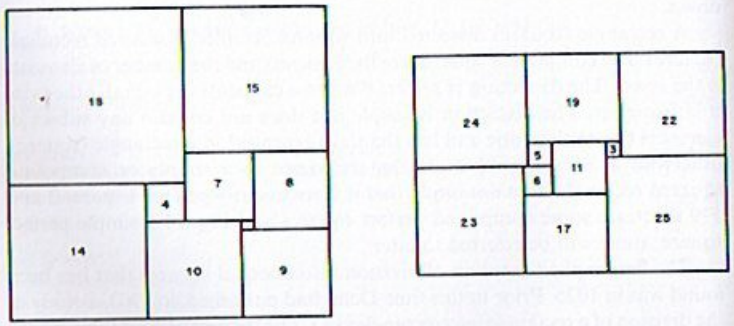

Figure 1 Moroń's two rectangles.

Moroń raises the question "For what rectangles is it possible to dissect them into squares?" He then observes "if there exists a rectangle (of different sides) for which there are two dissections R_1 and R_2 such that (1) in neither of these dissections does there appear a square equal to the smaller side of the rectangle and (2) each square of dissection R_1 is different from each square in dissection R_2, then the square is dissected into squares, all different, as shown in the following figure." Moroń's figure is reproduced here as Fig. 2. Either or both R_1 and R_2 could be compound, subject to Moroń's first condition.

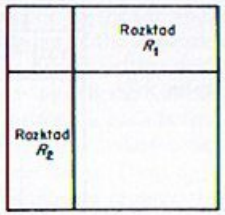

Figure 2 Moroń's figure.

Moroń's paper was practically unknown for some time. In 1935 the problem "Can a rectangle be divided into different squares?" appeared in a German journal. The answer, published in 1937, called attention to Moroń's paper and gave his two figures [10]. Steinhaus cites Moroń in the first edition (1938) of his *Mathematical Snapshots* [12]. In the meantime Kraitchik (1930) published the proposition, communicated to him by the Russian mathematician N. N. Lusin, that it was not possible to divide a square into a finite number of different squares [6]. A Japanese mathematician, M. Abe, was active in 1931–1932. His first paper, described under [8], was in Japanese and was unknown to other workers in the field until just a few years ago. His second paper [9] was in English and was known. It gives a simple perfect squared rectangle 195 × 191 and shows how an infinite series of compound squared rectangles can be built up from it with the ratio of sides approaching one in the limit.

Activity in Germany in 1937–1939 is shown by some problems [11, 19], a thesis [14], and three papers by Sprague. Sprague produced a perfect squared square, published in 1939 [15], the first one published. The square is made up by first forming two compound perfect squared rectangles with the same sides and then joining them as shown in Fig. 2. The structure of the square is shown in Fig. 3. Rectangles I and II are Moroń's two simple perfect rectangles; these are combined with an added 33 square and enlarged 29 times as shown

Figure 3 Structure of Sprague's square.

at the upper left. The same combination, magnified 13 times, is used again in the lower right. Rectangle III is a simple perfect rectangle of order 12 and sides 377 × 256 (number 633-b in [25] and 633-2 in [38]). The resultant compound perfect square has 55 elements and its side is 4205. Sprague had constructed a number of simple perfect rectangles in addition to the few already known (this fact is referred to in [14] and [48]) and the ingenuity of the construction of the square suggests numerous trials.

Additional results of the German activity are in Sprague [17, 18] and Reichardt and Toepken [19] and are commented on in the notes.

During the years 1936–1938, four students at Trinity College, Cambridge (R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte) were working on the problem. The resulting paper [21] is a classic. A highly interesting account of their research, by Tutte, appeared in 1958 [36]. This account should be read not only by those having some interest in the present problem but also for its own sake, as an adventure in mathematics.

At first the group constructed simple squared rectangles empirically [36]. How this was done is as follows. Draw a rough sketch of a rectangle divided into rectangles, subject only to the condition that no two of these rectangles have an entire side in common with each other, and then imagine that each of the rectangular elements is a square. The geometry of the figure and

elementary algebra enable the relative sizes of the assumed squares to be calculated. The result is a squared rectangle, perfect if none of the elements come out zero. This first stage was replaced by a theoretical treatment inspired by a wholly novel idea, which was the key to the topological aspects of the problem.

Only the highlights of the resultant paper [21] can be noted. The basic concept is the equating of a simple squared rectangle to an electrical network of a particular type. This is illustrated in Fig. 4. The squared rectangle on the left (1) is represented by the network on the right (2). Each horizontal line segment of (1) is represented by a node and each square is represented by a branch connecting the two nodes of the top and bottom lines of the square. The nodes a and b are the poles of the network and the resistance of each branch is unity. Ignoring the figures written in (2), if an unknown current is assumed entering at a and leaving at b, the Kirchhoff laws provide just enough equations to calculate the current in each branch in terms of the unknown current, the value of which is then taken so as to make the currents all integers without any common factor. These are the numbers written in (2), which are the sides of the component squares of (1). Thus electrical theory is brought into the picture and utilized, and graph theory as well, and the paper makes contributions to each.

The network of Fig. 4(2) was called a p-net (from polar net). If the two nodes a and b are connected by a new branch (in which the battery is located) the net is completed and was called a c-net. The c-nets are a special type of graph; they are all 3-connected planar graphs. The basic theorem demonstrated in the paper is that every simple rectangle can be derived from a c-net (3-connected planar graph) in the manner described. If the c-net has n edges, n p-nets are produced by removing each edge in turn, and hence n squared rectangles of order $n - 1$. The dual of a c-net results in the same rectangles, rotated 90°, and their p-nets are corresponding polar duals. The

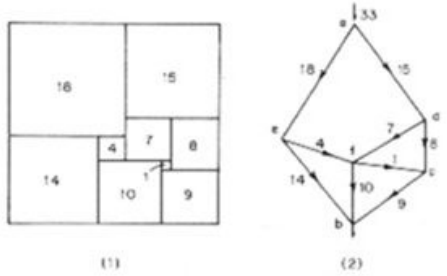

Figure 4 Squared rectangle and p-net.

process can also be thought of as placing a battery in turn in each edge of the c-net and calculating the relative values of the currents in the other edges, assumed to have unit resistance. Not every squared rectangle produced in this manner will be perfect (if the c-net has a symmetry, some rectangles might be imperfect with some of these still simple and others not), but every simple perfect rectangle of order $n-1$ is produced from the complete set of c-nets of order n. The authors produced all the simple perfect rectangles from orders 9 to 12 (they showed that there are none below 9) but gave a full description of only those of orders 9 and 10. The c-nets to order 13 had to be first constructed, of course.

One reason for constructing simple perfect rectangles was that one with equal sides might be encountered, thus being a perfect square. This did not happen. A theoretical method of constructing perfect squares, based on certain types of 3-pole networks which are to be combined, was developed and perfect squares produced. Five are mentioned (four compound and one simple) but only one, the lowest order obtained, is shown. The square is shown here in Fig. 5. Two others were described before paper [21] appeared, one in [16, 20], shown in Fig. 6, and the other in [20].

The treatment in the paper is graph theoretical and utilizes electrical theory as well. With respect to the latter, a proposition now known as the

Figure 5 Perfect square shown in [21].

Figure 6 Perfect square [16, 20].

matrix tree theorem appears for the first time (it was subsequently independently discovered by others, at least three times). Though latent in known electrical and matrix theory, it had not previously been made explicit and expressed, perhaps because a use for it had not yet appeared. The theorem expresses a certain square matrix, the first cofactors of which are equal in value and give the number of spanning trees of the connected graph from which the matrix is formed. This number, called the complexity of the graph in [21], plays a role in the theory of squared rectangles. If the incoming current in a p-net is taken as equal to the complexity (number of spanning trees) of the p-net, the values of all the currents in the wires work out as integers; the horizontal side of the resultant squared rectangle is equal to this complexity, and the semi-perimeter is equal to the complexity of the c-net from which the p-net was derived. If the elements as thus calculated have a common factor they and the rectangle are "reduced" by this factor. In the case of a squared square the complexity of its p-net is equal to kn^2, where n is the reduced side of the square.

The concluding section of the paper mentions a number of generalizations including rectangled rectangles and the dissection of polygons (of angles $\pi/2$ and $3\pi/2$) and cylinders. A 3-pole p-net would give a squared hexagon. For rectangulations the wires of the p-net have general resistances, not necessarily equal. Curiously, a professor of electrical engineering some time ago wrote several papers in which some electrical network problems were solved by translating the network into a rectangled rectangle (or polygon) and then treating the latter.

The first perfect square published, Sprague's square, appeared only a few months before the one in [16], and two incidental results in [21] were also anticipated in publication, in [17–19].

Smith and Tutte continued developing the theory of the nets producing squared rectangles, particularly the nets from which squared squares could be constructed [31]. In a companion paper [32] Tutte further developed the theory of constructing perfect squares and described a number of additional examples, including several simple squares.

C. J. Bouwkamp in Holland worked on the problem of squaring rectangles in 1945–1946. His paper, in three parts [25], presented the method of [21] described in a more physical manner and developed various properties of the nets. The c-nets with up to 14 edges were constructed and shown by drawings of each. All the simple squared rectangles of orders 9 to 13 were constructed and Part II of the paper lists them and their elements. A code was developed for describing a squaring by listing the elements (here referring to the lengths of the sides of the component squares) in a certain order; the Bouwkamp code is commonly used. This work reached the practical limit of what could be done by hand. Resumption by computer came later.

In 1948, T. H. Willcocks, an employee of the Bank of England at Bristol, a chess enthusiast, and an amateur mathematician, produced the lowest order

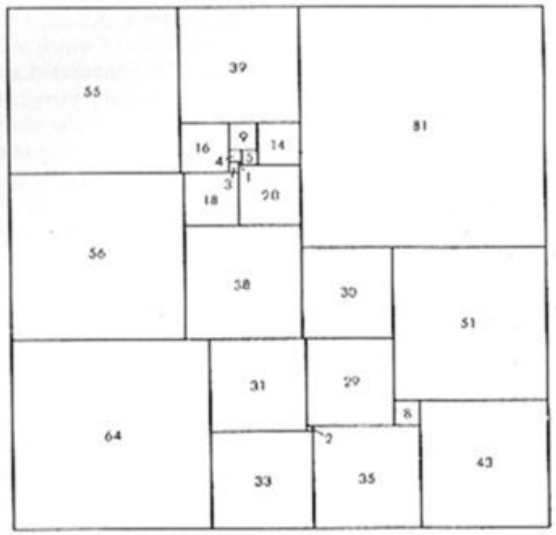

Figure 7 Lowest order compound perfect square known (24-175).

square known until 1978, shown in Fig. 7. He knew of the Brooks, Smith, Stone, and Tutte paper [21] but not the Bouwkamp paper [25]. Without the benefit of a catalogue of squared rectangles, he constructed his own stock and produced a number of compound perfect squares of low order. These he published in *The Fairy Chess Review*, a small periodical (now extinct) devoted to chess and other problems and puzzles [29]. His methods were based on the following. Referring to Fig. 2, if the corner square of R_2 which touches R_1 is removed and R_1 pushed in so that a corner fills the resultant space, another squared square can be produced which would be perfect if the smaller added square is not duplicated. This method is described [21] for a special case and the production of a perfect square of order 39 noted. Willcocks introduced two variations. First, he used a compound perfect rectangle, as shown in Fig. 8. Note that the two basic simple rectangles used are the same as those in Fig. 5 (in fact Fig. 8 can be made by cutting up Figure 5 to show this), and that three more perfect squares can be produced from them. The other variation was the utilization of imperfect squared rectangles. Since a corner element is to be discarded, this element could be the same size as some other element of the rectangle. Willcocks accordingly constructed a stock of squared rectangles with a corner element and an adjacent element

Figure 8 Another Willcocks perfect square.

the same size. This led to the order 24 perfect square [29]. If the squared rectangle and the elements 81 and 64 in Fig. 7 are removed it will be seen that the vacant corner of what is left can be filled with a square of side 30, forming the imperfect squared rectangle with two equal adjacent elements. A paper of 1950 [33] describes the above methods and various other methods utilized by Willcocks, as well as additional perfect squares he had constructed.

The Willcocks perfect square of order 24 was the lowest order perfect squared square known. Despite many attempts, a perfect square of lower order was not found until 1978, not even another one of order 24. Tutte has said "If the merit of a perfect square is measured by the smallness of its order, then the empirical method of cataloging the perfect rectangles had proven superior to our beautiful theoretical method" [36]. It is still the lowest order compound perfect square known.

In 1960 the Dutch group, Bouwkamp, Duijvestijn, and Medema, produced by computer and listed all the c-nets with up to 19 edges and all the simple squared rectangles with up to 15 elements. The former list, a few copies of which had been distributed, subsequently became generally available [39, 62]. Two catalogues of squared rectangles were published [37, 38], the later one evidently superseding the earlier one. The catalogue [38] lists the perfect squared rectangles (3663 in number) in two tables, one according to the c-nets from which they are derived and one according to the nondecreasing ratio of the short to the long side of the rectangle. The simple imperfect rectangles (431 in number) are also listed and the catalogue includes several other tables.

A deterrent to initiating computer work was the absence of a method of constructing all the c-nets with a given number of edges with certainty that there were no omissions. This problem was solved by a graph theory theorem of Tutte which he communicated to Bouwkamp before publication of a paper including it [42]. This theorem as used in the computer work depends upon the operation of forming a c-net with $n + 1$ edges from one with n edges by adding a new edge connecting two nonadjacent vertices of a mesh (face). Given the complete set of c-nets of order n, this operation is performed on each face (not applicable to a triangular face) of each c-net in all possible ways. The result is a collection of $n + 1$ order c-nets which includes every c-net of order $n + 1$ or its dual (with one exception); there will be a considerable amount of duplication. The graph known as a wheel (the description of which is apparent from this name), which has an even number of edges, is not produced by the method and must be added to the collection if $n + 1$ is even.

The method and the program are given in detail in Duijvestijn's thesis of 1962 [43]. This involved the problems of the representation of the graphs for use by the computer, generating new graphs from the old ones, generating

dual graphs, and testing the large numbers of graphs obtained to eliminate duplicates and equivalents.

The c-nets having been obtained, a program was also developed for generating and listing the squared rectangles derived from each c-net. Only those through order 15 were listed at this time; later, the perfect ones through order 18 were also listed [47]. Table I gives the number of simple perfect rectangles through order 18.

TABLE I

Number of Simple Perfect Rectangles

Order	Number	Order	Number
9	2	14	744
10	6	15	2609
11	22	16	9016
12	67	17	31427
13	213	18	110384

One reason for systematically generating squared rectangles, as has been stated, is that if the sides of any turn out to be equal, a squared square is obtained. Duijvestijn's thesis was also concerned with this problem. Squared rectangles produced from the c-nets through order 19 were tested for equal sides. The c-nets of order 20 were also generated, those whose complexity showed that a square was not possible eliminated, and the remainder utilized for generating the squared rectangles and testing for equal sides. No perfect square resulted; there were 101 simple imperfect squares of orders from 13 to 19. The computer time was great; a matter of 30 hours is mentioned in connection with the operations dealing with the c-nets of order 20. (See last paragraph.)

One by-product of this construction of complete sets of c-nets up to order 19 was the revival of Euler's problem of enumerating the combinatorially distinct convex polyhedra. As has been stated, the c-nets are 3-connected planar graphs. These are isomorphic with the graphs (vertices and edges) of convex polyhedra and it can be said that simple perfect squared rectangles are produced from convex polyhedra by means of Kirchhoff's laws relating to electric currents. The convex polyhedra with up to 19 edges were now known; previous results, which had not gone this high, could be checked, and further results produced. (An account of the enumeration problem is in [71].)

Tutte's popular article in the *Scientific American* and its reproduction in book form [36] inspired an unknown number of people to work on squaring rectangles and squares. One result was [45], which summarized known

methods of constructing compound perfect squares and introduced a new method. This was to first construct squares divided into unequal squares and one rectangle (deficient squared squares). These can be constructed easily; by the theoretical method of [21] simply by letting the resistance of one wire be unknown and imposing the condition that the resultant dissected rectangle have equal sides. This then fixes the unknown resistance, the value of which gives the ratio of the sides of the rectangular element. However, the paper used the patterns of the squared rectangles (perfect and imperfect) given in [38], by assuming one element to be a rectangle, the sides of the dissected rectangle to be equal, and then recalculating the elements, as being simpler for work by hand. Then a perfect rectangle, simple or compound, to fill the rectangular space was sought from the catalogue [38], which has a table arranged by ratio of sides. In this manner a large number of new compound perfect squares of low order (defined as 28 or less) were found, in addition to known ones. The lowest new ones were two of order 25, one of which is shown in Fig. 9; note that in this one a compound perfect rectangle was used. Duijvestijn, who had a prepublication copy of the paper, programmed the method for the computer but did not obtain any perfect square

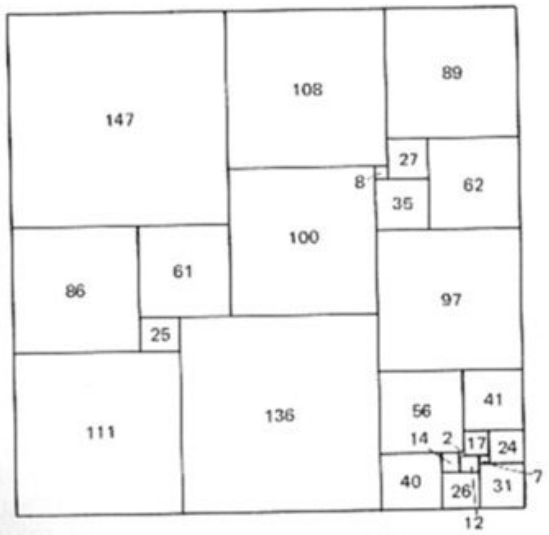

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Figure 9 Perfect square from [45].

under order 24 (private communication). The paper mentions deficient squared squares with two (or more) rectangles, but only one such perfect square of low order was found.

Compound perfect squares of medium or high order can be produced in profusion [51], by hand or by computer, and there is little interest now in merely multiplying the number. As has been stated, there have been attempts to lower Willcocks' record of the order 24 square, unsuccessful as to compound perfect squares. The latest attempt in this area is in the two papers of Kazarinoff and Weitzenkamp [67, 68], who show that there is no compound perfect square of order 21 or less. They used a graph theory approach and the treatment is too involved to attempt a brief description. A good deal of the work of the Dutch group, generating c-nets and squared rectangles by computer, was repeated, and additional work was involved in the procedure for searching for compound squared squares. No results of perfect squares actually produced are mentioned. A number of simple perfect rectangles of reduced side ratio p/q with $p + q < 30$ are described; there is one of order 17 with side ratio 3/5 and one with ratio 5/7.

The basic paper of Brooks, Smith, Stone, and Tutte tackled the problem of simple perfect squares. These, as has been indicated, do not include a squared subrectangle in their structure. A highly theoretical method was developed, utilizing 3-pole nets satisfying certain strict conditions, by means of which a simple perfect square of order 55 (side 5468) was produced [21, 26]. Brooks managed to obtain a simple perfect square of order 38 (side 4920), described in [32] and illustrated in [34]. By a clever modification of the manner of connecting the basic nets, Willcocks reduced this square to one of order 37 (side 1947), described in [43] and [45]. Relaxation of the conditions and utilization of some rearrangements enabled the production of simple perfect squares down to order 31 [53a, 54], but this is probably the lowest order possible for this particular theoretical method.

The construction of simple perfect squares is the subject of Wilson's thesis [56]. A novel method is used which is explained by the simplest example. Two p-nets are connected in parallel, that is, the poles of one are made to coincide with the poles of the other, forming a new p-net with the number of wires the sum of the number in each. The complexity of the new p-net is calculated (the unreduced horizontal side of the rectangle from this p-net) and also the potential drop between the poles (the unreduced vertical side of the rectangle); these two numbers are to be coprime. The complexity is written in the form kn^2, where k is an integer free of square factors. Since n is to become the reduced side of any squared square produced, it is to be greater than 53 as it was known that any simple perfect square must have at least 20 elements. The key requirement is that there must be a pair of vertices a and b such that the potential drop between them is divisible by kn. Pairs

of p-nets satisfying the above requirements were sought by computer. The basic p-nets used were those for the perfect squared rectangles up to order 15 in Bouwkamp's catalogue [38], excluding those having a reduction factor.

In each of the admissible new p-nets, the vertices a and b are taken as poles and a squared rectangle formed from it with these poles. Its horizontal side (unreduced) will be kn^2 (the number of spanning trees of the graph is independent of which vertices are to be designated as poles). The theory developed by Wilson from graph theory considerations shows that the vertical side of this squared rectangle will be equal to the horizontal side, or some multiple of it (no multiples occurred in the work).

In this manner Wilson obtained 5 simple perfect squares of order 25, and 24 of order 26. It may be noted that each of these can be separated into two squared hexagons, which follows from the method of construction, as is apparent in Fig. 10, reproduced from the thesis.

Other combinations of two basic p-nets, with 1, 2, or 3 added wires, were utilized, and also different orientations of the basic p-nets. Any combinations which would have more than 26 wires were excluded initially. An incidental result was the production of simple squared square cylinders (height equals

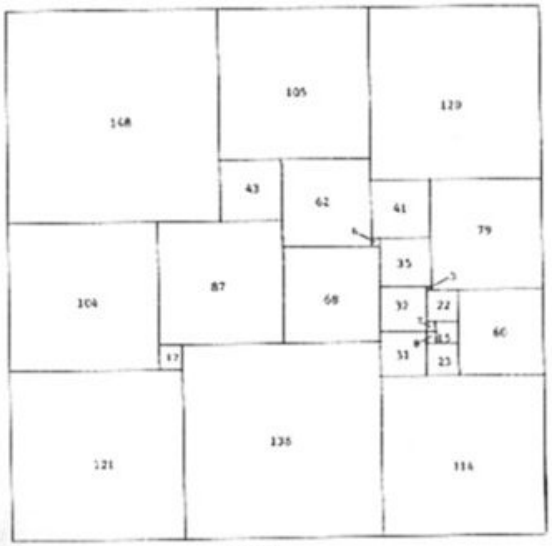

Figure 10 One of Wilson's five simple perfect squares of order 25.

circumference). If the two new poles a and b are not part of the same mesh, the result is a squared cylinder rather than a squared square, as indicated in [21]. Thirty-one simple perfect squared cylinders were found, of orders 24, 25, and 26.

Table II gives the number of perfect squares of order 31 or less known up to 1978. The 25 and 26 order simple squares included Wilson's 5 and

TABLE II

Number of Known Perfect Squares to Order 31 (1977)

Order	Simple	Cmpd 1	Cmpd 2
24		1	
25	8	2	
26	28	10	1
27	6	19	
28		33	4
29		49	1
30		19	14
31	4	36	1

24, respectively; the others, as well as the 27 order ones, were derived from some of these by a method of partial dissections and recombinations of portions (described in [53a]). The number of compound perfect squares, with one (Cmpd 1) or two (Cmpd 2) subrectangles, are from [51]; the note indicates the sources. The 26 order one with two subrectangles (Fig. 5) is probably the lowest order with two subrectangles possible. No squares with three subrectangles appear in the table as the lowest order one found is of order 38. A simple perfect square of order 21 was produced in 1978 [73].

The present status of the subject of dissecting rectangles and squares into unequal squares is indicated by the review which has been made, in the text along with the Bibliography.

As to general theory, we are very little beyond what was developed by Brooks, Smith, Stone, and Tutte in their series of papers.

As to simple perfect rectangles, it has not yet been shown that *any* given rectangle with commensurate sides can be divided into unequal squares in a simple manner. The general method of producing simple perfect rectangles could be characterized as building up rectangles from unequal squares, as originally put by Moroń. If a simple perfect rectangle with a given ratio of sides does not appear in the systematic building up process to the order that it has been carried, then special methods for the particular case need to be devised, as in the case of simple squares and the 2×1 rectangle [63, 64]. The results obtained would necessarily appear in the systematic process if carried

further to a sufficiently high order. Proof of the general proposition, which is indicated as plausible in [21], would have to be theoretical.

With respect to compound perfect squares which depend upon availability of perfect rectangles for their construction, it has been positively shown that there are none below order 22. The work of the Dutch group may have bettered this result, but it has not been assembled and described so as to constitute any proof of a minimum. It remains to be proven whether or not there are any compound perfect squares of orders 22 and 23.

Several methods of producing simple perfect squares by computer have been devised. Wilson's method has been described; he introduced some unnecessary limitations to shorten the work and hence it cannot be said that his method is incapable of producing simple perfect squares below order 25. Reference [64] mentions but does not describe a computer program used for producing simple squares; no perfect ones below order 25 were found by this method (and another unpublished program), probably because of failure to include a sufficient range of data. What might be called the brute force method, constructing all the simple perfect squared rectangles up to a given order and testing them for equality of sides, was carried out by Duijvestijn through order 19 in 1962 [43]. Reference [70] indicates the development of faster methods and an intention to carry on the work beyond order 19. This was done, with improvements in the methods, and a simple perfect square of order 21 produced in March 1978 [73]. The square is shown on the cover of this book. There are none below order 21 and only one of order 21. Duijvestijn is now in process of deriving order 22 simple perfect squares but results have not yet been published.

Note This paper was delivered and in press before publication of Duijvestijn's order 21 square. Changes reflecting this fact were made in the proof but had to be kept to a minimum. New references [35a, 58a, 69a, 71, 72, 73] have been added.

ACKNOWLEDGMENTS

I am indebted to Michael Goldberg for making the drawings for Figures 3, 5, 6, and 8. Figures 4 and 9 are reproduced from [71] and [45], respectively, and Figure 7 from Martin Gardner's book [36]; it was gazing at this particular figure that gave the idea for [45].

BIBLIOGRAPHY

The following bibliography is a list of papers and other items concerned with dividing a rectangle or a square into unequal squares, with a few collateral items. There is an attempt at completeness (seldom successful and the author would appreciate being advised of any omissions) with respect to material making some contribution to the subject, substantial or otherwise. Review articles are also included and some other material, either germane to the historical

account or having some point of interest. While a few works on mathematical recreations are listed, in general such works and other material which merely show a known squared rectangle or square have been excluded. Nearly all the items have a note, particularly those not mentioned in the text and in some other instances adding to what is said in the text.

1. Max Dehn, Über die Zerlegung von Rechtecke in Rechtecke. *Math. Ann.* **57** (1903) 314–332.
See text for comment. Cited in [5, 7, 10, 11, 14, 17, 21, 35, 40, 43, 58, 68].

2. Sam Loyd, "Cyclopedia of Puzzles," Privately published, 1914 (not seen); The Patch Quilt Puzzle, reprinted *in* "Mathematical Puzzles of Sam Loyd, Selected and Edited by Martin Gardner," Vol. 1, pp. 73, 147–148. Dover, New York, 1959.
A square quilt made of 169 square patches of the same size is to be divided into the smallest number of square pieces by cutting along lattice lines. The answer, 11 squares with sides 1, 1, 2, 2, 2, 3, 3, 4, 6, 7, is neither perfect (equal squares) nor simple (adjoining equal squares form a subrectangle). Gardner states that this puzzle first appeared in 1907 in a puzzle magazine edited by Sam Loyd. Referred to and illustrated in [6].

3. Henry E. Dudeney, Puzzle 173, Mrs. Perkin's Quilt, "Amusements in Mathematics," pp. 47, 180. London, 1917; reprinted, Dover, New York, 1958. Puzzle reprinted *in* Dudeney, "536 Puzzles & Curious Problems," (Martin Gardener, ed.), pp. 120, 324–325. Dover, New York, 1967.
Same problem as in [2]. See comments by the editor and in [53].

4. Henry E. Dudeney, "The Canterbury Puzzles and Other Curious Puzzles," 1st ed. London, 1907 (not seen); 4th ed., 1919; reprinted, Dover, New York, 1958.
Puzzle 40, pp. 66–67, 191–193, requires the dissection of a given square into 12 unequal squares and one rectangle, the dimensions of the latter being given. This is the puzzle referred to by Tutte in [36] as sparking some of the work there described.

5. Zbigniew Moroń, O rozkładach prostokatow na kwadraty (On the dissection of a rectangle into squares). *Przegląd Mat. Fiz.* **3** (1925) 152–153.
See text. A manuscript English language translation (by Michael Goldberg) of this Polish paper is available from the author. Cited in [10, 12, 21, 25, 35, 43, 55, 56, 58, 66, 68, 69].
I am indebted to Dr. Stanisław Dobrzycki of Lublin, Poland, for obtaining information concerning Moroń. The following is quoted from a letter to him by Prof. Władysław Orlicz:

> "Zbigniew Moroń was my younger schoolmate when studying mathematics at the University of Lwów; about 1923–24 we were both junior assistants in the Institute of Mathematics. Professor Stanisław Ruziewicz (who was then professor of mathematics at the University) communicated to us the problem of the dissection of a rectangle into squares. He had heard of it from the mathematicians of the University Kraków [Cracow] who took interest in it. As young men we enthusiastically engaged ourselves in investigating this problem, but after some time we all came to the conclusion that it was certainly as difficult as many other apparently simple questions in number theory. The examples found by Moroń were to us a great surprise. Before the World War II Moroń was a teacher in secondary schools; after it he was too, and dwelt in Wrocław, where he died some 5 years ago."

Moroń was born in 1904 and died in 1971. Dr. Dobrzycki has translated the following from a later paper by Moroń (No. 35 or 35A):

> "In the years 1925–28 I found further results in this domain; among others I proved that it is impossible to construct a rectangle with less than 9 different squares; I also

knew of the dissection of a square which was later given by Sprague [No. 15]. Nevertheless I did not publish them, but only exposed them at meetings of the mathematical seminar of Professor Ruziewicz."

6. M. Kraitchik, "La Mathématique des Jeux ou Récréations Mathématiques," p. 272. Stevens Frères, Brussels, 1930.
After giving the figure of Sam Loyd's puzzle [2] Kraitchik states, "But it is not possible to decompose a given square into a finite number of squares no two equal. This proposition though not demonstrated appears to be true. It was communicated to us by Monsieur Lusin, professor at Moscow." The next edition, in English (1942), gives an example (from [21]) and states that the problem was "long thought to be impossible." Cited in [14, 20, 21, 25, 40, 58, 68].
7. A. Schoenflies and M. Dehn, Ungelöste Probleme der analytischen Geometrie, *in* "Einführung in the analytische Geometrie der Ebene und des Raumes," 2nd ed., Appendix VI, pp. 402–411. Springer-Verlag, Berlin, 1931.
Includes parts of [1]. Cited in [10, 14, 21, 25].
8. Michio Abe, On the problem to cover simply and without gap the inside of a square with a finite number of squares which are all different from one another (in Japanese), *Proc. Phys.-Math. Soc. Japan* **4** (1931) 359–366.
This paper, in Japanese, was unknown to workers in the field until just a few years ago and hence is not treated in the text. The results are surprising for the time. Abe attempted to obtain a perfect square by the method of combining two simple squared rectangles of the same size. He constructed over 600 simple perfect rectangles, of which six are described by illustrations. While he found one pair the same size, two 608 × 407 rectangles of order 13, the resultant squared square was imperfect. His total for rectangles of order 13 was not complete and he missed the pair which gives perfect results. He also tried another method, noted under [45], but again with no success. His manner of drawing the squared rectangles suggests that he may have known Moron's paper [5]. Cited in [9].
9. Michio Abe, On the problem to cover simply and without gap the inside of a square with a finite number of squares which are all different from one another (in English), *Proc. Phys.-Math. Soc. Japan* (3) **14** (1932) 385–387.
See text. Applications of the method are in [20] and [49]. Cited in [14, 17, 21, 35, 58].
10. Jaremkewycz, Mahrenholz, and Sprague, Answer to Problem 1242, *Z. Math. Naturwiss. Unterricht Schulgattungen* **68** (1937) 43; Proposed in **66** (1935) 251.
See text. Cited in [14, 15, 21, 58].
11. H. Toepken, Problem 242, *Jber. Deutsch. Math.-Verein.* **47**, Part 2 (1937), p. 2.
Relates to [1]. Cited in [14, 21].
12. H. Steinhaus, "Mathematical Snapshots," 1st ed., pp. 8, 9, 131. Steckert, New York, Leipzig, 1938.
Gives assembling a rectangle from nine given unequal squares as a problem, citing Moron for the solution. Cited in [13, 20, 21, 58].
13. S. Chowla, Division of a rectangle into unequal squares, and Problem 1779, *Math. Student* **7** (1939) 69–70, 80.
Solves the trivial problem posed by Steinhaus [12]. A separate item, Problem 1779, asks "It is possible to divide the volume of a rectangular parallelopiped into unequal cubes?" An elegant proof of the impossibility (quoted or paraphrased many times, e.g., in [59]) is given in [21]. Cited in [21, 58, 68].
14. Alfred Stöhr, Über Zerlegungen von Rechtecken in inkogruente Quadrate, *Schr. Math. Inst. Inst. Angew. Math. Univ. Berlin* **4**, Part 5 (1939) 118–140.
Dissertation, University of Berlin, Berlin, 1938. Cited in [15, 17, 21, 35, 40, 58].

15. R. Sprague, Beispiel einer Zerlegung des Quadrats in lauter verschiedene Quadrate, *Math.* **45** (1939) 607–608.
See text. Cited in [17, 18, 20, 21, 32, 35, 36, 40, 41, 45, 48, 50, 52, 55, 56, 58, 66–69].
16. A. H. Stone, Problem E401, *Amer. Math. Monthly* **47** (Jan. 1940) 48.
The first part of this problem lists the sides of 28 different squares and asks that they be fitted together to make a single square of side 1015. This is in effect a publication of the perfect squared square (illustrated in Fig. 5) since the assembling is trivial. The second part asks if there is any simpler perfect squared square. Cited in [20 (solution); 21, 32, 58].
17. R. Sprague, Über die Zerlegung von Rechtecken in lauter verschiedene Quadrate, *J. Reine angew. Math.* **182** (1940) 60–64.
See [18]. Cited in [18, 21, 32, 34, 35, 40, 58, 68].
18. R. Sprague, Zur Abschätzung der Mindestzahl inkongruenter Quadrate, die ein gegebenes Rechteck ausfüllen, *Math. Z.* **46** (1940) 460–471.
Sprague proves the theorem that any given rectangle with commensurate sides can be dissected into unequal squares (in a compound manner). The method requires initial knowledge of two perfect squared rectangles of the same size with no two squares alike, which he gives in [15], and is explained here by this example. The ratio of the sides of each squared rectangle of [15], call them R_1 and R_1', is 13 to 16. Magnify R_1 by 16 and R_1' by 13 and join the two by the now equal short side of one and the long side of the other, forming squared rectangle R_2 with 53 elements and five subrectangles, and sides 61,625 × 30,160. Now magnify R_1 by 13 and R_1' by 16 and form squared rectangle R_2'. R_2 and R_2' are the same size and have no two like elements and form perfect square S_2 which has 108 elements and ten subrectangles. S_1 (the perfect square of [15]) and S_2 do not have any like elements when brought to the same size. In a similar manner square S_3 (with 214 elements and 20 subrectangles) is formed from R_2 and R_2', and so on, forming a series of perfect squares with no like elements when brought to the same size. The rectangle to be dissected, with sides m and n, say, is divided into mn unit squares and these are filled with mn perfect squares from the series. The result would be very highly compounded, a 2 × 3 rectangle, for example, dissected in this manner would appear to have 3351 elements and contain 315 subrectangles. A different and more elegant proof, utilizing graph theory, but still resulting in very high compounding, is in [21]. Cited in [21, 32, 34, 35, 40, 58].
19. H. Reichardt and H. Toepken, Solution to Problem 271, *Jber. Deutsch. Math. Verein.* **50**, Part 2 (1940), pp. 13, 14; (a) Problem posed in **48**, Part 2 (1939) p. 73.
Shows that there is no simple perfect rectangle less than order 9 and gives the two of order 9. A different proof is in [21]. Cited in [21 (a only); 58].
20. Michael Goldberg and W. T. Tutte, Solution of Problem E401, *Amer. Math. Monthly* **47** (Oct. 1940) 570–572.
See [16]. Goldberg solves the first part and gives some discussion. Tutte solves the second part by referring to the 26 order perfect square to appear in the forthcoming paper (Fig. 6); he also describes another perfect square of order 28 by listing its elements. There is an added editorial note. Cited in [21, 33, 34, 36, 45, 58].
21. R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, The dissection of rectangles into squares, *Duke Math. J.* **7** (1940) 312–340.
See text. Also referred to in the notes to [6, 13, 18–20, 54, 59]. Cited in [25–27, 30–38, 40, 41, 44–46, 48, 50, 52, 55–58, 61, 63, 65–69].
22. Science Service, Science in 1940, Mathematics, *Science* **92**, Suppl. 14 (1940).
Lists the division of a square into a finite number of smaller squares no two the same size as one of the achievements of 1940. Cited in [34, 45].
23. H. Toepken, Problem 294, *Jber. Deutsch. Math. Verein.* **51**, Part 2 (1941) 2.

24. Michael Goldberg, Squaring the square, *Washington Scientist* **1** (1945) 76.
Discussion of problem and report of a talk by A. H. Stone before the Philosophical Society of Washington, March 31, 1945.

25. C. J. Bouwkamp, On the dissection of rectangles into squares, Paper I, *Koninkl. Nederl. Akad. Wetensch. Proc. Ser. A* **49** (1946) 1172–1188; Papers II and III, **50** (1947) 58–71, 72–78.
See text. Cited in [26, 27, 30–34, 36–38, 40, 44–46, 54, 58, 61, 62, 68].

26. R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, A simple perfect square, *Koninkl. Nederl. Akad. Wetensch. Proc. Ser. A* **50** (1947) 1300–1301.
Gives the details of the first simple perfect square, of order 55, referred to in [21]. Cited in [32, 36, 45, 55, 58].

27. C. J. Bouwkamp, On the construction of simple perfect squared squares, *Koninkl. Nederl. Akad. Wetensch. Proc. Ser. A* **50** (1947) 1296–1299.
Revises Section 8 of [25]; gives the details of another simple perfect square, of order 55; corrects a few minor errors in [25]. Cited in [32, 36, 45, 58, 68].

28. T. H. Willcocks, Problem 7523 and solution, *Fairy Chess Rev.* **6** (1947–1948) 114 (Dec.); 123 (Feb.).
Rediscovery of the perfect square of [16, 20]. Cited in [29].

29. T. H. Willcocks, Problem 7795 and solution, *Fairy Chess Rev.* **7** (1948) 97 (Aug.); 106 (Oct.)
See text. Cited in [32, 33, 36, 41, 43, 45, 48, 50, 52, 55, 57, 58, 66–68].

30. W. T. Tutte, A note on a paper by C. J. Bouwkamp, *Koninkl. Nederl. Akad. Wetensch. Proc. Ser. A* **51** (1948) 280–228.
Comment on a point in [25, III].

31. C. A. B. Smith and W. T. Tutte, A class of self-dual maps, *Canad. J. Math.* **2** (1950) 179–196.
See text. Cited in [32, 34, 36, 56, 58].

32. W. T. Tutte, Squaring the square, *Canad. J. Math.* **2** (1950) 197–209.
See text. Cited in [33, 34, 36, 40, 45, 46, 52, 54, 56, 58, 61, 63, 67, 68].

33. T. H. Willcocks, A note on some perfect squared squares, *Canad. J. Math.* **3** (1951) 304–308.
See text. Cited in [41, 45, 46, 48, 49, 52, 54–56, 58, 67–69].

34. Michael Goldberg, The squaring of developable surfaces, *Scripta Math.* **18** (1952) 17–24.
General discussion with nontrivial examples of squared cylinders and cones, and a Möbius strip. Cited in [40, 56, 58].

35. Zbigniew Moroń, Orozkładach prostokatow na nierowne kwadraty (in Polish), *Wiadom. Mat.* (2) **1** (1955–1956) 75–94.
Extensive review article (not read). Cited in [50].

35a. Zbigniew Moroń, O prawie doskonalych rozkładach prostokatow (On almost perfect dissections of rectangles). *Wiadom. Mat.* (2) 1 (1955–1956) 175–179.
Not seen.

36. W. T. Tutte, Squaring the square, *Sci. Amer.* **199** (1958) 136–142, 166. Reprinted with addendum and enlarged bibliography in Martin Gardner, "The 2nd Scientific American Book of Mathematical Puzzles and Diversions," pp. 186–209, 250. Simon & Schuster, New York, 1961; also paperback edition.
See text. Cited in [45, 52, 54, 57, 60, 61, 64, 67, 68].

37. C. J. Bouwkamp, A. J. W. Duijvestijn, and P. Medema, "Catalogue of Simple Squared Rectangles of Orders Nine through Fourteen and their Elements." Technische Hogeschool, Eindhoven, Netherlands, May 1960, 50 pages.
See text. Cited in [36, 38, 43, 45].

38. C. J. Bouwkamp, A. J. W. Duijvestijn, and P. Medema, "Tables Relating to Simple Squared Rectangles of Orders Nine through Fifteen." Technische Hogeschool, Eindhoven, Netherlands, Aug. 1960, 360 pages.
See text. Reviewed in [41]. Cited in [36, 41–45, 48, 49, 52, 55, 56, 62, 65–69].

39. C. J. Bouwkamp, A. J. W. Duijvestijn, and P. Medema, Table of c-nets of Orders 8–19 Inclusive, 2 vols. Philips Research Laboratories, Eindhoven, Netherlands, 1960; unpublished, available in UMT file of Mathematics of Computation.
See text. Described in [62].

40. Herbert Meschkowski, Die Zerlegung von Rechtecken in inkongruente Quadrate, *in* "Ungelöste und unlösbare Probleme der Geometrie," pp. 92–103. Vieweg and Sohn, Braunschweig, 1960. English translation, The decomposition of rectangles into Incongruent Squares, *in* "Unsolved and Unsolvable Problems in Geometry," pp. 91–102, 164–165. Oliver and Boyd, Edinburgh and London, 1966.
Cited in [50, 58, 61].

41. Michael Goldberg, Review of [38]. *Math. Comp.* **15** (1961) 315.
Cited in [45].

42. W. T. Tutte, A theory of 3-connected Graphs, *Koninkl. Nederl. Akad. Wetensch. Proc. Ser. A* **64** (1961) 441–455.
See text. Cited in [43, 48, 52, 56, 68].

43. A. J. W. Duijvestijn, Electronic computation of squared rectangles. Dissertation, Technische Hogeschool, Eindhoven, Netherlands, 1962; also in *Philips Res. Rep.* **17** (1962) 523–612.
See text. Cited in [45, 48, 52, 54–56, 62, 67, 68, 70].

44. W. T. Tutte, A census of planar maps, *Canad. J. Math.* **15** (1963) 249–271.
Develops formulas for calculating the number of "rooted" c-nets which can also be used for estimating the number of simple perfect rectangles with a given number of elements, also described in [48].

45. P. J. Federico, Note on some low-order perfect squared squares, *Canad. J. Math.* **15** (1963) 350–362.
See text. Gives the full description of 35 low order squares, 24 of them new. (The method, described in the text, turns out not to be entirely new as Abe in the unknown Japanese paper [8] distorted some perfect rectangles to form a squared square with one rectangle, but obtained no results as he only applied the method to perfect rectangles with sides differing by one.) Cited in [48, 49, 52, 55, 56, 67–69].

46. S. L. Basin, Generalized Fibonacci sequences and squared rectangles, *Amer. Math. Monthly* **70** (1963) 372–379.
A series of squares with sides according to the Fibonacci series can be arranged in a rectangle; it is compound and also imperfect as it has two adjoining equal squares of side one. Cited in [49, 58].

47. C. J. Bouwkamp, A. J. W. Duijvestijn, and J. Haubich, Catalogue of Simple Perfect Squared Rectangles of Orders 9 through 18, Philips Research Laboratories, Eindhoven, Netherlands, 1964 (unpublished, 12 vol., 3090 pp. listing 154,490 simple squared rectangles).
See text. It would take over 6000 volumes the same size to list the simple perfect rectangles of orders 19 to 23. Cited in [62, 65].

48. W. T. Tutte, Squared rectangles, *Proc. IBM Sci. Symp. Combinatorial Problems, Thomas J. Watson Research Center, Yorktown Heights, N.Y.* (1964) 3–9.
Review article with discussion, announces and illustrates Wilson's first 25 order simple perfect square (see [56]). Cited in [68].

49. P. J. Federico, A Fibonacci perfect squared square, *Amer. Math. Monthly* **71** (1964) 404–406.
A Fibonacci rectangle (see [46]) is found with which a compound perfect square is formed by removing the corner unit square and combining with a simple perfect rectangle. Cited in [56].

50. L'Udovit Vittek, A perfect decomposition of the square into 25 squares (in Slovak), *Mat. Fyz. Casopis* **14** (1964) 234–235.
Rediscovery of a perfect square published in [45].

51. P. J. Federico, List of Perfect Squared Squares, 119 pp., May 1964; Supplement, 40 pp., Aug. 1971.
Manuscript catalogue of over 800 compound perfect squares; includes published squares, and unpublished ones derived by Bouwkamp, Federico, Willcocks, and E. Lainez. Listed here, as copies were distributed. (Lainez is a Spanish metallurgical engineer who read the French translation of [36] and, without the aid of a catalogue of rectangles, constructed his own stock and derived a number of new perfect squares of low order, including the first one in which the subrectangle is necessarily on a side of the square instead of in a corner.)

52. W. T. Tutte, The Quest of the Perfect Square, *Amer. Math. Monthly* **72**, Part II (1965) 29–35.
Review article with discussion. Cited in [54, 60, 61, 67–69].

53. Martin Gardner, Mrs. Perkins' Quilt and other square-packing problems, reprinted with additions from *Sci. Amer.* **215** (1966) 264–272, **216** (Jan. 1967) 118–121; in Martin Gardner, "Mathematical Carnival," pp. 139–149, 272. Knopf, New York, 1975.
Includes the problem: Can 24 squares with sides 1 to 24 (the sum of whose squares equals 70^2) be assembled into a square of side 70?

53a. P. J. Federico, Note on Some Simple Perfect Squares, 22 pp., March 1967.
Manuscript paper; listed here as copies were distributed and results are referred to in [54] and [69].

54. T. H. Willcocks, Some squared squares and rectangles, *J. Combinatorial Theory* **3** (1967) 54–56.
Tutte says in [69], "The two last-mentioned authors (Federico and Willcocks) have modified the theoretical method (of [21]) to obtain simple perfect squares of orders ranging down to 31." Cited in [55].

55. W. T. Tutte, Topics in graph theory, *in* "Graph Theory and Theoretical Physics," (F. Harary, ed.), pp. 301–312. Academic Press, New York, 1967.

56. John C. Wilson, A method for finding simple perfect squared squarings, Ph.D. Thesis, University of Waterloo, Waterloo, Ontario, 1967 (80 pp. plus 72 pp. computer output).
See text.

57. G. H. Morley, Networks and squared squares, *Eureka* (*J. Archimedeans, Cambridge Univ. Math. Soc.*) No. 30, Oct. (1967) 14–16.
A highly ingenious method of constructing simple perfect squares of high order, 60 or above, from specially related pairs of squared rectangles, discovered by the author while in high school.

58. I. M. Yaglom, "How to Divide a Square" (in Russian). Nauka, Moscow, 1968.
The first and only book on the subject; contains some new material (not read). The bibliography includes three Russian items (not seen) which appear to merely show a known perfect square. Cited in [68].

58a. P. J. Van Albada, La dissection du carré en carrés, *Bull. Soc. Math. Belg.* **20** (1968) 161–170.

59. Mark Kac and Stanislaw M. Ulam, "Mathematics and Logic," p. 32. Mentor paperback, New York, 1969.
The electrical analogy for [21]) cited as an "illustration of the remarkable and wholly unexpected connections of which mathematics is full."

60. Sherman K. Stein, "Mathematics, the Man-Made Universe," 2nd ed. Freeman, San Francisco, California, 1969; 3rd ed., 1975.
Chapters 7 and 8 give an interesting elementary exposition of "tiling" a rectangle with unequal squares. Cited in [61].

61. Ross Honsberger, Squaring the square, *in* "Ingenuity in Mathematics," New Mathematical Library, No. 23, pp. 46–60. Random House, New York; now by Math. Assoc. Amer., Washington, D.C., 1970.

62. C. J. Bouwkamp, Review of [39], *Math. Comp.* **24** (1970) 995–997.

63. R. L. Brooks, A procedure for dissecting a rectangle into squares, and an example for the rectangle whose sides are in the ratio 2 : 1, *J. Combinatorial Theory* **8** (1970) 232–243.
First solution of the problem "to find a simple perfect rectangle whose horizontal side is twice the vertical side" [36]. The solution has 1323 elements. Cited in [64, 68, 69].

64. P. J. Federico, Some simple perfect 2×1 rectangles, *J. Combinatorial Theory* **8** (1970) 244–246.
Second and further solutions of the problem of [63]; 23, 24, and 25 order 2×1 simple perfect rectangles are shown. Cited in [67–69].

64a. R. D. Hollands, Developing a problem, *Math. in Teaching* (*Assoc. Teaching Aids in Math.*) No. 50, Spring (1970) 64–66.
Elementary introduction from standpoint of problem solving and teaching.

65. C. J. Bouwkamp, On some special squared rectangles, *J. Combinational Theory Ser. B* **10** (1971) 206–211.
Discusses simple perfect rectangles having the property that the set of elements can be arranged in two different ways forming two dissections; lists 59 examples. Cited in [68].

66. Howard Eves, "A Survey of Geometry," Rev. ed., Vol. 1, pp. 229–231. Allyn & Bacon, Boston, Massachusetts, 1972; 1st ed., 1963.
Listed to show the inclusion of the subject in a textbook on geometry.

67. N. D. Kazarinoff and R. Weitzenkamp, On existence of compound perfect squared squares of low order, *J. Combinatorial Theory Ser. B* **14** (1973) 163–179.
See text. Cited in [68].

68. N. D. Kazarinoff and R. Weitzenkamp, Squaring rectangles and squares, *Amer. Math. Monthly* **80** (1973) 877–888.
See text.

69. W. W. Rouse Ball and H. S. M. Coxeter, "Mathematical Recreations and Essays," 12th ed. Univ. of Toronto Press, Toronto, 1974; also in paperback.
Section on squared rectangles and squares contributed by W. T. Tutte.

69a. Lionel March and Philip Steadman, Electrical networks and mosaics of rectangles, *in* "The Geometry of Environment," pp. 263–283. MIT Press, Cambridge, Massachusetts, 1974.
Application to architectural design.

70. A. J. W. Duijvestijn, Fast calculation of inverse matrices occurring in squared-rectangle calculation, *Philips Res. Rep.* **30** (1975) 329–339.

71. P. J. Federico, The number of polyhedra, *Philips Res. Rep.* **30** (1975) 220*–231*.
Application to the enumeration of polyhedra. Figure 4 is reproduced from this article.

72. Albert A. Mullin, on arithmetic aspects of geometric problems. *Am. Math. Soc. Notices*, **25** (Feb. 1978) A-227.
States several "Lemmas" on the nature of the integers forming the side-lengths of the elements of a perfect squared rectangle. One of these is that the sides cannot all be in arithmetic progression. This settles the problem mentioned under [53].
73. A. J. W. Duijvestijn, Simple perfect squared square of lowest order. *J. Combinatorial Theory*, *B* **25** (1978) 260–263; See also *Sci. Amer.* (June 1978) 86–87.
See last paragraph of text.

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Courtesy J. A. Bondy

W. T. Tutte